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THE STONE SKELETON

JACQUES HEYMAN
Department of Engineering, University of Cambridge

Abstract—The mode of action of masonry construction is investigated, using the principles of plastic design developed originally for steel frames. These principles are applied to the analysis of the structural system of the Gothic cathedral; the flying buttress and the nave vault are treated in some detail.

INTRODUCTION

FITCHEN [1] speaks of "... the total lack of written documentation on both the engineering structure and the erectional procedures..." of Gothic constructions. Modern work has been, in general, descriptive rather than analytical in the engineering sense. Two nineteenth-century German writers, in the course of lengthy works on the techniques of stone construction, undertake some structural analysis. Breymann [2] has a clear understanding of the behaviour of masonry, and Ungewitter [3] gives a general account of the complete structural action of a Gothic cathedral. A little later, Guadet [4] made some calculations on vaults and flying buttresses; these are reported and expanded by Rosenberg [5]. Frankl [6] has made what appears to be an almost complete study of the literature, but his interests are mainly aesthetic and historic rather than technical.

Of contemporary writings, Suger's [7] are clearly not technical, and the Album of Wilars de Honecort [8] appears to be a collection of private memoranda, and infuriatingly assumes that the reader already knows how to build a cathedral.

Fitchen has gone a long way in reconstructing building techniques that must have been used by mediaeval master-builders; in the course of his studies, Fitchen is forced to consider also the structural system. It is the purpose of the present paper to consider some of the engineering aspects of stone construction, and to apply the general principles evolved to the Gothic skeleton. One or two elements (the flying buttress, stone vaults) will be treated in some detail.

Fitchen's book on the construction of Gothic cathedrals is basic to this study, and a plain reference here to Fitchen refers to the book. Similarly, Viollet-le-Duc's [9] "Dictionnaire raisonné..." will be referred to simply as Viollet-le-Duc.

While very little is known of the strictly engineering techniques of mediaeval master-builders, studies of Wilars and other manuscripts have uncovered some of the "mysteries" of the masons' lodges. Such reconstructed rules of building are entirely numerical, and deal with the practical determination of $\sqrt{2}$, the relative proportion that one part of a building should bear to another, the automatic determination of elevations from plans, and so on. It is almost certain that the builders were incapable of even the simplest structural analysis. One of the classic mediaeval problems was that of the parallelogram of forces, not solved until the end of the sixteenth century; without any rules for the composition of forces, or, indeed, any clear formulation of the notion of a force and of its line of action (Straub [10]), it is difficult to see how any calculations can have been done to determine, for example, the line of thrust in a buttress.

There can, however, be no doubt of the practical capabilities of the master-builders; a cathedral which survives almost intact for 800 years is clearly a work of genius. Equally clearly, the structural system employed, measured by almost any yardstick, is almost perfect. Certainly advances must have been made by trial and error, by experiments with the actual structure as well as with models. But, looking at, for example, the complete glass curtain-walls of the Sainte-Chapelle in Paris, one is tempted to sense a mastery of building technique greater than any that can be ascribed to mere trial and error.

This mastery was maintained with complete authority for only a very short period, in a very limited region: for about 144 years in France. In 1140 work started on the new choir of the abbey church of St. Denis; as Pevsner [11] says: "Whoever designed the choir of St. Denis, one can safely say, invented the Gothic style." The collapse of Beauvais in 1284 signalled the end of the greatest period of cathedral building, and subsequent work never achieves the structural quality of that of the thirteenth century.

It is true that Gothic continued to develop for a further two hundred years, and that isolated peaks were reached, not only aesthetically, but also structurally (e.g. fan vaults in England). By and large, however, there was a general and gradual debasement of the structure, until the whole system was swept away by the Renaissance, and the mysteries of the craft guilds of the Middle Ages were forgotten.

THE MATERIAL

Stone seems a good constructional material for, say, mountains, but it is not clear that it is equally suitable, except possibly in the form of caves, for buildings. Indeed, typical Romanesque structures are cavernous; their aim is to disperse forces as much as possible (for example, by means of domes), and it is with extreme caution that a Romanesque wall is pierced to admit light. By contrast, the aim of the Gothic system is the concentration of forces; a cathedral was required to be as tall and as light as possible.

Viollet-le-Duc says (article "Arc-boutant") that from the moment that the flying buttress was clearly expressed in the building, the structure of the cathedral developed. "To ask for a Gothic cathedral without flying buttresses is to ask for a ship without a keel." He continues: "Noting correctly that a well-buttressed vault needs only a small vertical supporting force at its springing, compared with its weight, the builders little by little made the piers more slender, and carried the thrust to the outside, to the main buttresses. They opened completely the spaces between the piers, beneath the formerets, by great mullioned windows, and the whole structural system of the great naves is reduced to slender piers made rigid by the load, and kept in vertical equilibrium as a consequence of the balance achieved by the vault thrusts and the counter thrusts of the flying buttresses." The description is clear; forces are collected by means of vaults and arches into slender columns and buttresses, and the walls become non-structural sheets of glass.

Later, in his article "Pilier", Viollet-le-Duc writes: "The column is too slender to withstand on its own inclined thrusts; to stay vertical, it must be loaded vertically." Here the basic action of a masonry element is clearly expressed; applied loading must be resisted purely by axial thrust, and bending is inadmissible. The mediaeval builders understood this basic requirement of their designs; when they forgot that masonry cannot take bending, as with the flying buttresses at Amiens (see below), a bad design resulted.

Assuming for the moment, then, that bending of any stone structural element is absent, it is convenient at this point to enquire as to the stress levels likely to be encountered in a Gothic cathedral. Yvon Villarceau [12] used a significant parameter: the height to which a prismatic column could be built before crushing at the base due to its own weight. The calculation is simple; a medium sandstone of weight say 150 lb/ft^3 and crushing strength say 6000 lb/in^2 (Ungewitter gives 4300 to 12,800 lb/in^2) can be erected, in the absence of instability, to a height of over a mile.

Now the stresses in a cathedral are almost entirely due to the vertical dead weight of the material; even allowing what might be called a load concentration factor for the columns, of value say 4, to allow for the weight of the roof and other parts of the structure which are not self-supporting, a cathedral would have to be say 1,500 ft high before the columns started crushing. (It is assumed that a certain minimum slenderness ratio for the columns has been established to prevent instability.)

Yvon Villarceau advocates a factor $\frac{1}{10}$ on the height of his self-crushing column; i.e. the stresses (nominal) should be limited to $\frac{1}{10}$ of the crushing stress of the material. Thus a cathedral might be built, at this stress level, to a height of say 150 ft; Westminster Abbey and Notre Dame, Paris are about 110 and 120 ft to the soffit of the timber roof, Beauvais about 170 ft.

It is clear that if nominal working stresses are limited to such low values, the question of the *strength* of the masonry can be expected to hardly enter the calculations. Instead it is likely that individual structural elements, and the structure as a whole, must be proportioned on the basis of stability. A flying buttress will not, in general, fail by crushing, but may, if badly proportioned, buckle; similarly, the mode of failure of a main buttress will be one of overturning about its base.

The next stage in the enquiry, therefore, must be to examine the behaviour of masonry constructions under conditions of low stress.

THE STABILITY OF MASONRY CONSTRUCTION

Fitchen notes that "... the curve of pressure must remain within the middle third (or, at most, the middle half) of the thickness of the structure". This statement of the middle-third rule carries, within parentheses, its own contradiction. Adherence to the middle-third rule ensures that tension is not developed, or, rather, that the material is everywhere in compression. If this is a proper design requirement, it is difficult to see how the relaxation, from middle third to middle half, can be permitted. Pippard and Baker [13] make a similar comment during their discussion of the behaviour of voussoir arches; concentration (from the time of Galileo onwards) on the elastic behaviour of structures has ensured the survival of rules like the middle-third.

Moseley [14], in a standard text, states in fact quite clearly that the requirement is that the line of thrust should not pass outside the entire cross-section, and this was equally clearly understood by Coulomb [15]. Failure of a masonry structure will occur when the line of thrust can no longer be contained within the stonework.

This statement, which can be made only as a result of certain simplifying assumptions, will be elaborated later. It is the basic stability statement for the stone skeleton.

Coulomb's paper, in which he also lays the foundations of the science of soil mechanics, discusses the behaviour of stone structures. Coulomb's Fig. 14 (Fig. 1 here) shows half of a masonry arch, and he considers the failure of the arch under the action of its own weight and of the horizontal thrust, say H, acting at the crown through the point f.

Taking first a material (like soil) having both friction and cohesion (shear strength), Coulomb determines the value of H necessary to cause failure of the arch by sliding along the plane Mm. Failure of the arch is thus dependent on the material properties. However, Coulomb remarks immediately that "friction is often so high for the materials used in arch construction, that the voussoirs can never slide one on another". He then concludes that if failure is to occur at section Mm, it must be by rotation of a portion of the arch about either m or M.

If φ is the weight of the portion Ga Mm of the arch, acting in the line g′g, it will be seen that if rotation is taking place about M,

$$H = \varphi \frac{\mathrm{gM}}{\mathrm{MQ}} \tag{1}$$

and, if about m,

$$H = \varphi \frac{\mathrm{qg}'}{\mathrm{mq}} \tag{2}$$

Now these expressions for H involve the density of the material and the dimensions of the arch, but do not involve the strength of the material. The expressions are thus purely geometrical statements of the stability of the arch.

Coulomb shows that the maximum value of H must be sought from (1), by considering various cross-sections of failure and lines of action of H; similarly, the minimum value must be found from (2). This sort of calculation is made below for flying buttresses; it may be remarked (as Coulomb remarks) that a method of trial and error is very accurate, since the maximum or minimum is very "flat".

Coulomb also notes that the line of thrust cannot be supposed to pass exactly through points like G or M, since this would lead to infinite stresses, and crushing must occur as the thrust line approaches either the extrados or the intrados. However, errors in calculations will be small if a generally low stress level is postulated. Thus, if the nominal axial thrust is such that the nominal stress is $\frac{1}{10}$ of the crushing stress, the line of thrust may approach the edge of the masonry to within 5 per cent of the depth of the section.

LIMIT DESIGN PRINCIPLES

For the purpose of establishing general principles and theorems, the following assumptions will be made about the properties of the material:

(i) *Stone has no tensile strength.* This assumption is almost exactly true if a structure, like an arch, is under consideration, made up of voussoirs laid either dry or with very weak mortar. Although the stone itself may in fact have some tensile strength, the joints will not, and no tensile forces can be transmitted from one portion of the structure to another. The assumption of no tensile strength is, in accordance both with common sense and with the corollaries of the general principles established below, a safe assumption. It may be slightly too safe if the stone structure is not of the voussoir type, and if tensile forces can be transmitted by the interlocking of perhaps randomly oriented stones.

(ii) The general stress levels are so low that, for the purposes of calculation, *the compressive strength of stone is effectively infinite.* This is a slightly unsafe assumption, and will be discussed more fully later.

(iii) *Sliding of one stone upon another cannot occur.* This seems a reasonable assumption (it did to Coulomb, as noted above). It implies that wherever there is a weak plane, for example between voussoirs, the line of thrust should not depart too far from normality to that plane. (The collapse of Beauvais may have been initiated by sliding.)

With those assumptions, Coulomb's proposed mode of failure by *hinging* at a free edge, is the only possible mode. Under these conditions, Kooharian [16] (see also Prager [17]), has shown that masonry may be treated as a material to which the limit theorems, developed for the analysis of the plastic behaviour of steel frames, may be applied. (See, for example, Baker *et al.* [18].) Failure occurs when, in the words of Pippard and Baker [13], sufficient hinges are "... formed to transform the structure into a mechanism".

Consider a portion of a masonry structure, formed from voussoirs, and whose local depth is $2h$, Fig. 2(a). If a hinge is just forming, as shown, under an axial load N, then referred to the centre line of the masonry, the hinge forms at an effective bending moment $M = hN$. In Fig. 2(b), the lines OA, OB are $M = \pm hN$; any point *within* the open triangle AOB represents a state of the cross-section under consideration which is safe, i.e. the line of thrust lies wholly within the masonry.

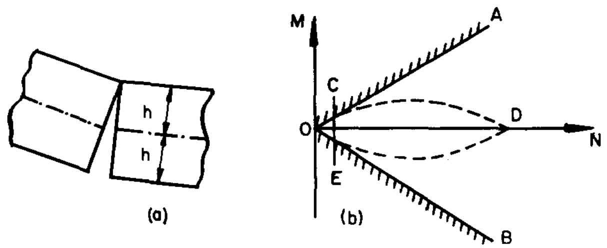

FIG. 2. Limit condition for masonry voussoirs.

A point in Fig. 2(b) lying on OA or OB represents the formation of a hinge; the line of thrust lies in one of the surfaces of the masonry. A point *outside* the triangle AOB represents an impossible state; the line of thrust is outside the masonry.

If the stone of infinite strength is replaced by the real stone with a definite crushing strength, the *yield surface* AOB is replaced by the curved boundary OCDEO, formed by two parabolic arcs. The same considerations enumerated above apply; a point within the new yield surface represents a safe state, and so on.

If now nominal stresses are limited to say 10 per cent of the crushing strength, the portion of the yield surface under consideration is the slightly curvilinear triangle OCE; OC and OE are so nearly straight that little error will be introduced by taking them to coincide with OA and OB.

Thus although in fact the line of thrust cannot approach a free edge to within closer than 5 per cent of the depth of the section, it will be assumed that hinging occurs as in Fig. 2(a). As noted above, this assumption is slightly unsafe.

Accepting these postulates of the material behaviour, the *uniqueness theorem* may be stated for masonry as follows: If a line of thrust can be found which represents an equilibrium state for the structure under the action of the given external loads, which lies wholly within the masonry, and which allows the formation of sufficient hinges to transform the structure into a mechanism, then the structure is on the point of collapse; further, if the loads can all be specified as ratios of one of their number (proportional loading), and the loads have been notionally increased from their working values to the collapse values by a load factor, then the value of that load factor at collapse is unique.

The "*safe*" *theorem* may be stated as follows: If a line of thrust can be found which is in equilibrium with the external loads and which lies wholly within the masonry, then the structure is safe. In this theorem, of the utmost importance to this study, it should be noted that the mechanism requirement of the uniqueness theorem has been dropped. It should be noted also that the line of thrust found in order to satisfy the safe theorem need not be the actual line of thrust; any line in equilibrium with the external loads and lying within the masonry is sufficient to ensure stability. Such an equilibrium solution is termed "statically admissible".

SOME DIGRESSIONS

Before making detailed calculations on specific structures, it is of interest to elaborate the principles of limit design by some general examples.

(a) THEOREM. *If, on striking the centering for a flying buttress, that buttress stands for* 5 *min, then it will stand for* 500 *years.*

This statement assumes that the loads on the flying buttress are static, and due to self-weight plus any thrusts from the nave vaulting. Thus the theorem cannot apply to the upper flying buttress in large Gothic cathedrals, whose function Fitchen [19] has demonstrated to be to resist wind forces acting on the great roof.

For static loads, however, the proof of the theorem follows immediately from the "safe" theorem. The fact that the buttress stands for 5 min is complete experimental evidence that a statically admissible thrust line can be found to lie within the stonework; the upper limit of 500 years depends, of course, on the decay of the material.

(b) Gifford and Taylor [20], reporting on the restoration of ancient buildings, say that "... an interesting structural curiosity exists at the east end of the retro choir in Wells Cathedral, in that the two flying buttresses on the east face of the east wall and window are carried eccentrically by two very slender columns from the retro choir. In the first instance the load is carried by the vault, which surprisingly shows no sign of distress".

Here again, the fact that no signs of distress have been observed is ample evidence that a satisfactory thrust line can be found.

(c) Poncelet [21] speaks of the safety factor (coëfficient de stabilité) needed for protection, among other things, against failure (défaut) of the foundations of a masonry construction. Now clearly the foundations themselves must be *adequate*, and for this purpose they must presumably be designed with some margin of safety. But if a nave pier, for example, settles differentially by say 1 in., what margin must be allowed in the *masonry* design to allow for such settlement?

THEOREM. *If the foundations of a stone structure are liable to small movements, such movements will never, of themselves, promote collapse of the structure.*

Thus no margin is necessary in masonry design to allow for small settlements or spreading of foundations.

It has been seen that stability of a stone structure will be ensured if a thrust line can be found in equilibrium with the applied loads and lying wholly within the masonry. Suppose that such a thrust line has been determined for the ideal structure with rigid foundations; that is, it is supposed that the structure is *originally* satisfactory, and that it has discovered for itself a statically admissible equilibrium state.

Consider now the effect of small movements of the foundations. Here "small" is being used in the usual structural sense; the movements are small compared with the leading structural dimensions. More precisely, equilibrium equations written for the original undeformed structure are valid for the deformed structure.

Now a thrust line is purely a graphical representation of equilibrium equations; if the thrust line is valid for the undeformed structure, it will also, by definition, be valid for the structure after small settlements have occurred. That is, the original statically admissible state of equilibrium will be statically admissible for the deformed structure.

Foundation settlements will, of course, lead to cracking of the masonry, or to the opening of voussoirs. Such cracking is, however, as will be seen below, the *normal state* of masonry; for small settlements the cracks may be hairline, or closed by virtue of the elasticity of the stone, and hence invisible. But such cracks, representing as they do merely the inability of masonry to carry tension, are not of themselves dangerous.

(d) Suger [7] gives a non-eyewitness account of a storm which arose during the building of the abbey church of St. Denis. The following extract is from the translation by Panofsky [22].

"...when the work on the new addition with its capital and upper arches was being carried forward to the peak of its height, but the main arches—standing by themselves—were not yet held together, as it were, by the bulk of the vaults, there suddenly arose a terrible and almost unbearable storm....

"...such a force of contrary gales hurled itself against the aforesaid arches, not supported by any scaffolding nor resting on any props, that they threatened baneful ruin at any moment, miserably trembling and, as it were, swaying hither and thither".

Fitchen identifies the "upper arches" with the nave vault and the "main arches" with the flying buttresses. Thus the nave walls must be pictured as tied together by the completed great timber roof, the flying buttresses completed and their centering struck, but the stabilizing shell of the nave vault not yet in place. In these circumstances the storm, if Suger is to be believed, caused trembling and swaying of the entire fabric.

In terms of the ideas in this paper the wind loads produced movements and perhaps visible cracking at various portions of the structure. The cracking indicated the formation of hinge points, and was, no doubt, alarming. However, hinges did not occur in sufficient numbers to transform the structure into a mechanism of collapse.

THE VOUSSOIR ARCH

The arch may be analysed as a particularly simple form of masonry construction; having established the general pattern of behaviour, the results will be applied to other structural elements.

Pippard *et al.* [23] and Pippard and Ashby [24] are among the modern investigators of the voussoir arch; their results are summarized by Pippard and Baker [13]. One conclusion, that collapse occurs when sufficient hinges are formed to give a mechanism, has already been quoted. Figure 3, based on one of Pippard and Baker's figures, shows such a mechanism, with four hinges, for the arch. Mechanistically viewed, the structure is now a four-bar chain, capable of deformation with one degree of freedom.

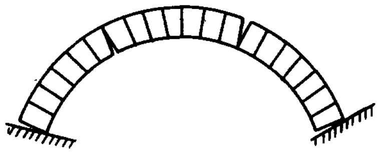

FIG. 3. Collapse of a voussoir arch (after Pippard and Baker).

The number of hinges required for a collapse mechanism is readily calculable. The arch, for example, viewed from the standpoint of conventional structural analysis, is a structure having three basic redundancies; three hinges in such a structure will make it statically determinate. The fourth, extra, hinge turns the determinate structure into a mechanism, and enables the value of the collapse load to be found (see e.g. Baker *et al.* [18] or Heyman [25]).

In fact, care must be taken when referring to the degree of redundancy of a masonry structure; this point is taken up below where it is shown that, in a sense, a masonry structure is always statically determinate. However, the conclusions as to the number of hinges required at collapse remain unaffected.

The idea of a mechanism of collapse is quite old. Frézier [26] quotes tests made by Danisy in 1732 (work reported to the Académie de Montpellier) and his Figs. 235–240, reproduced here in Fig. 4, give clear modes of collapse recorded from model experiments. Figure 239, for example, is a four-bar chain with hinges at the sections EF, GI, ef and ml. Note (a) the correct pattern of alternate opening and closing hinges (cf. Fig. 3), and (b) the double hinge at ef. This double hinge is clearly observable in tests, and was, for example, reported by Pippard *et al.*, and is due to the tangential position of the thrust line. The double hinge leads to the insensitivity of calculations to choice of hinge position which was noted by Coulomb.

The correlation of hinge positions with the position of the thrust line, i.e. the inter-dependence of equilibrium and mechanism conditions, was noted formally by, for example, Breymann [2]. He observes that, at a hinge, the line of thrust must pass through the hinge point. Further, if the hinge is at an internal cross-section, and not at the springing of an arch, then, since the line of thrust cannot pass outside the masonry, it must be tangential to the arch profile at the hinge point.

Figure 5 reproduces Breymann's Plate 85; he says: "It therefore follows that of all the lines of thrust that can be drawn, that which has the largest rise and smallest span must necessarily touch the crown of the arch near the centre and the intrados near the springing; it has the trace $\beta'\gamma'\alpha'$ in Fig. 7, Plate 85, and gives the minimum horizontal thrust."

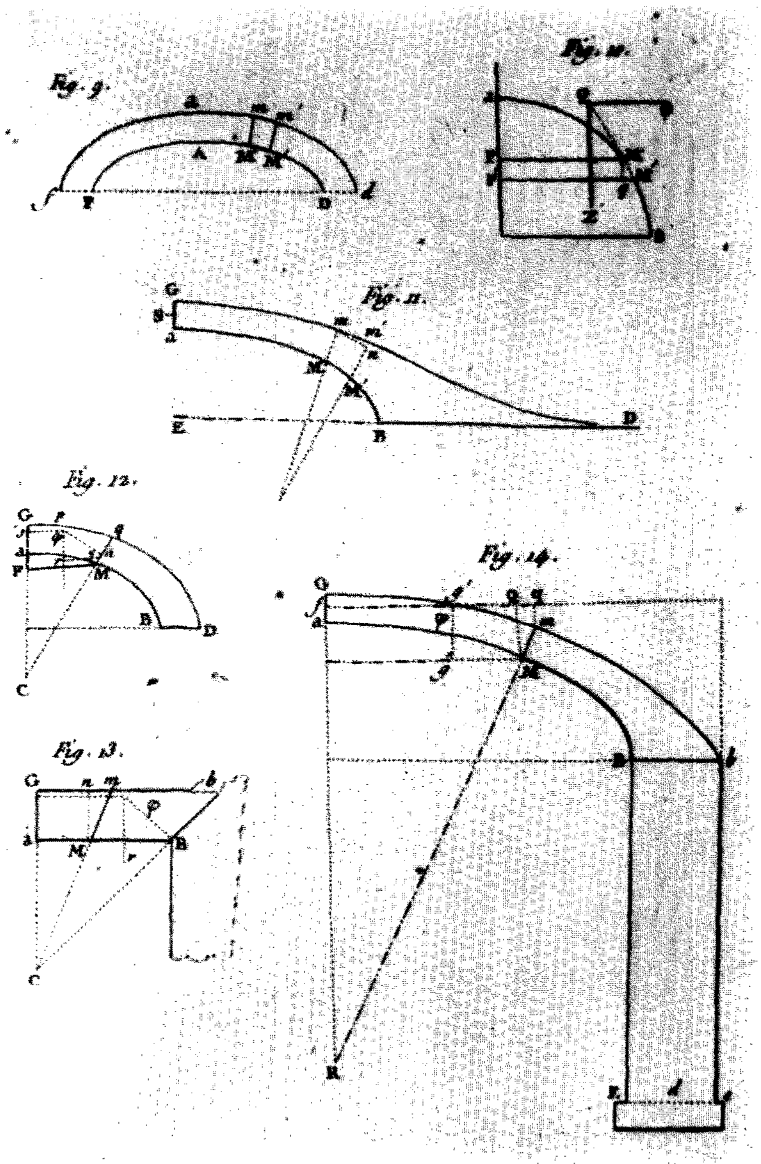

FIG. 1. Coulomb's Figs. 9–14; masonry arches.

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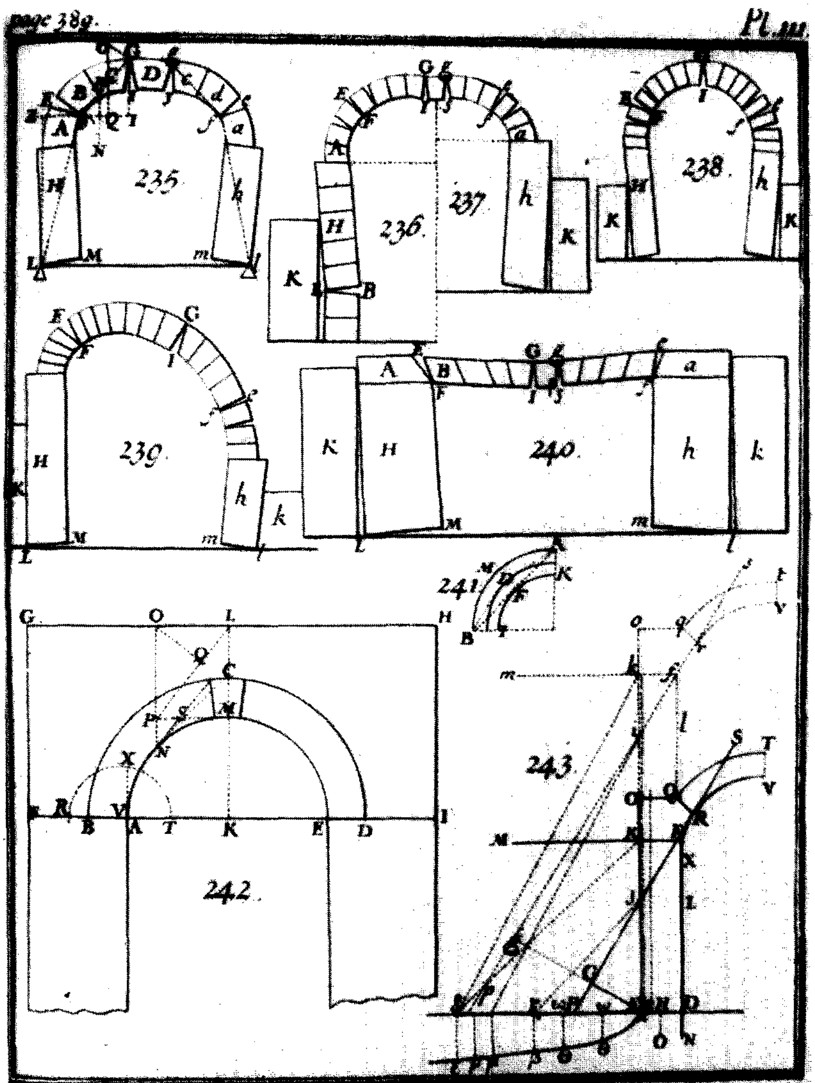

FIG. 4. Frézier's Figs. 235–243; collapse of arches.

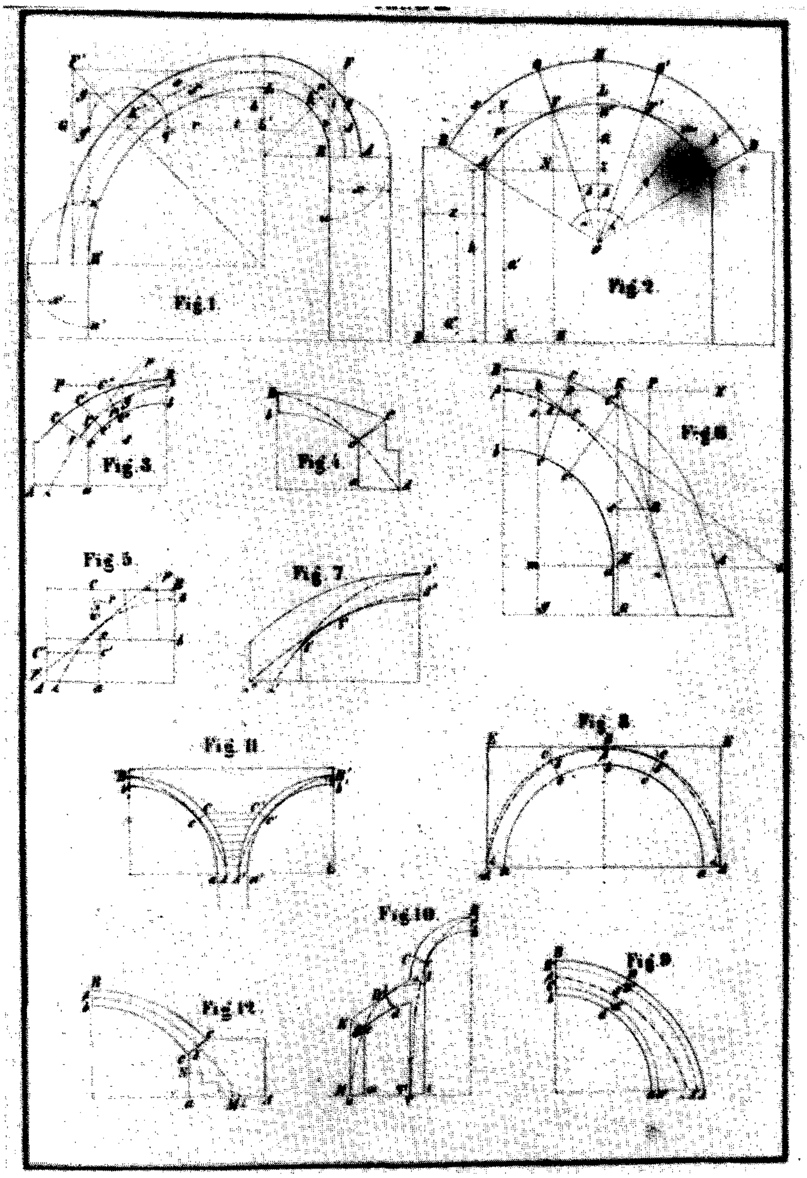

FIG. 5. Breymann's Plate 85; thrust lines.

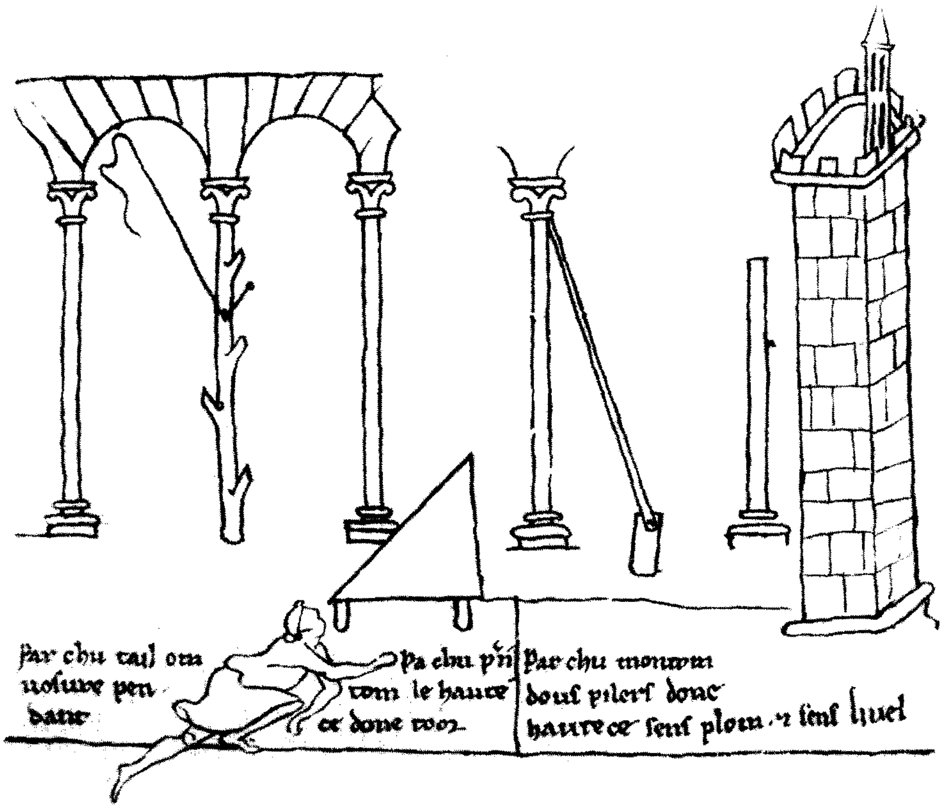

FIG. 8. Part of a page of the Album of Wilars de Honecort.

There are two important ideas here. First, there is a *choice* of lines of thrust. Secondly, each position of the line of thrust corresponds to a definite value of the horizontal thrust, of which the minimum is given by the line $\beta'\gamma'\alpha'$ in Fig. 7 (Fig. 5 here). It will be seen that the other line $\beta''\gamma''\alpha''$ sketched in the figure corresponds to the maximum horizontal thrust, and all possible lines of thrust (for the given dead weight loading) must lie between these two limits.

Now if the line of thrust passes through a point such as β' on the surface of the arch, that point is an incipient hinge. Thus, if the line of thrust is $\beta'\gamma'\alpha'$ together with its symmetrical reflection for the other half of the arch, there are three incipient hinges, and the arch is just statically determinate.

Breymann sketches in his Fig. 10 (Fig. 5 here) a buttressed circular arch on the point of collapse. The number of hinges (6) is correct, and the hinges alternately open and close. Further, the lines of thrust (despite some confusion at their point of intersection) touch the surface of the masonry at the hinge points.

Quite generally, the problem of collapse analysis of masonry construction consists in finding a line of thrust which passes through a number of hinge points sufficient to transform the structure into a mechanism.

It is pertinent to enquire as to the type of structure for which it is *impossible* to find such a line of thrust. One such is the plate-bande, illustrated for example by Frézier's Fig. 240 (Fig. 4). The collapse state shown involves tilting of the piers AH, ah, with a corresponding spread of the span. If the piers are rigid, such that no such spread can occur, then it will be seen that no combination of hinges, in the horizontal upper and lower surfaces of the plate-bande, will give rise to a mechanism. Similarly, considering a weightless arch, of which half is shown in Fig. 241 (Fig. 4), Frézier remarks that no mechanism can be formed with hinges at A and B if the intrados lies wholly to the right of the straight line AB. An arch of this type can therefore carry an indefinitely large vertical load at the crown A.

Frézier reaches this conclusion from purely mechanistic considerations, but an argument based upon the line of thrust is equally valid. For the vertical load carried at A, it is possible to find a statically admissible line of thrust, namely AB, which lies wholly within the arch; by the safe theorem, collapse cannot therefore occur.

The arch of Fig. 241 is, of course, incapable of carrying a vertical load placed *anywhere* on the span. A vertical load at M, for example, will induce a line of thrust which, in order to reach the right hand springing, must pass outside the masonry (Fig. 6).

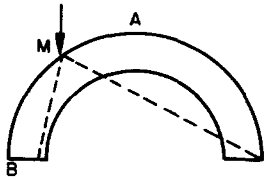

FIG. 6. Weightless arch (after Frézier).

There are numerous arches capable of carrying any combination of vertical loads without collapse, and Fig. 7 illustrates three based on the plate-bande. In Fig. 7(a) the intrados has been cut away, but a portion of each springing can be "seen" from every point on the extrados. Figure 7(b) shows an inclined plate-bande forming, essentially, a flying buttress type of structure; this will be analysed more fully later. Figure 7(c) is developed from Fig. 7(b) by again cutting away the intrados.

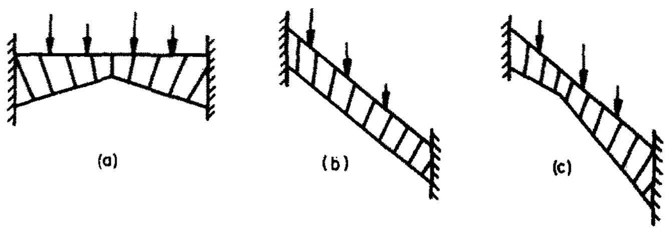

FIG. 7. Arches based on the plate-bande.

It is clear that the essential stability of masonry arches, if shaped to the proper profile, was well understood by mediaeval master builders. Indeed, such was their mastery that they could introduce humour into their constructions. Figure 8 reproduces one of Wilars' illustrations (see Willis [27] or Hahnloser [28]). It will be seen that after the arcade has been completed, the tree trunk may be removed. Torroja [29] gives further examples of mediaeval jokes of this sort.

THE STABLE STATE OF MASONRY

To summarize the work so far, the collapse (unstable) state of a masonry construction is characterized by the formation of hinges sufficient in number to turn the structure into a mechanism. The position of the thrust line is determined at the hinge sections, and the thrust line is unique at collapse for the collapsing portion of the structure.

If the structure is not collapsing, but is in a stable state under the given external loading, there is, in general, a wide choice of possible positions of the thrust line. Poncelet [21], in his review of arch structures, concludes that there is no satisfactory way of determining the actual position of the thrust line, and that further progress can be made theoretically only by considering the elasticity of the masonry. That is, Poncelet places the masonry arch quite firmly into the category of statically indeterminate structure, for whose solution the equations of compatibility and the stress–strain law are required.

Yvon Villarceau [12], at the start of a long Mémoire, states equally firmly that the theory of indeterminate structures is not the tool to use for arch design. He considers that masonry will not obey the idealized assumptions made in the analysis, and, indeed, voices much the same objections to elastic analysis that finally drove steel designers to develop plastic theory.

Instead, Yvon Villarceau replaces an actual arch, Fig. 9(a), by an idealized arch, Fig. 9(b), in which the voussoirs touch only on the centre line. He remarks, with absolute correctness, that if his method is satisfactory for the substituted idealized arch, then it will patently be satisfactory for the real arch.

A statically admissible thrust line is found, in fact, which coincides exactly with the centre line, and which is thus as far as possible from the surface of the real arch. The method is inverse; from a given loading, Yvon Villarceau deduces the shape that the arch must have for his design assumption to be satisfied. He notes that his designs will have a high resistance to accidental overload and to travelling loads.

Effectively, therefore, Yvon Villarceau satisfies only the equilibrium condition, and does not attempt to find the actual line of thrust in the arch as built. *This is sound limit design procedure, and leads to a quick and economical method of design.* Yvon Villarceau's tables given in his Mémoire are easy to use, and have never been superseded.

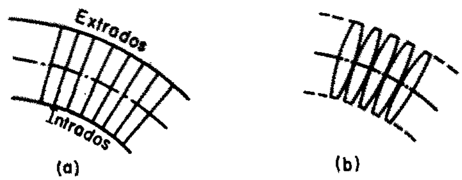

FIG. 9. Voussoir arch (after Yvon Villarceau).

The calculation of an actual line of thrust, using the theory of indeterminate structures as suggested by Poncelet, is in fact useless. To apply energy theorems, for example, implies the use of certain compatibility conditions, and these must be assumed at the start of the calculations. For example, it will be assumed for an arch that the springings neither spread nor rotate, both extremely unlikely events.

Pippard *et al.* made careful tests of voussoir arches, and concluded that if there were any practical imperfections, an apparently redundant arch turned itself into a statically determinate structure. For example, "... the fixed-end arch, if there is any movement of the abutments, becomes in effect a three-pinned arch" (Pippard and Baker). The reason is clear; neglecting compressibility of the stone, the shape of the arch is fixed by the shape of individual voussoirs. Now if this shape does not match *exactly* the shape implied by the geometry of the abutments, the arch will accommodate itself to those abutments by the formation of hinges. Figure 10 illustrates schematically the two cases of spread at the abutments and of the abutments being too close together; in either case the arch has three hinges and hence is statically determinate.

FIG. 10. Imperfectly fitted arches (after Pippard *et al.*).

Pippard *et al.* concluded that such hinges were always present, even if the voussoirs were so well fitted that they could not be observed. The natural state of incompressible masonry is, therefore, the statically determinate state, for which the thrust line is determined uniquely.

The actual elasticity of stone will, in practice, close up some of the cracks, and introduce a degree of indeterminacy into the construction. Any movement of the supports will, however, tend to convert the redundant structure to the determinate state.

In particular, the structure, closely fitted on its centering, will sag when that centering is struck, and in many cases cracks will appear immediately from which the hinge positions can be detected. Such cracks are completely harmless; one more hinge, at least, must be formed to turn the structure into a mechanism, and the geometry may be such that that extra hinge can never be formed.

THE FLYING BUTTRESS

Ungewitter [3] studied the action of flying buttresses, and his Table 41 comprising Figs. 402–410 is reproduced here as Fig. 11. Figure 402, for example, gives two possible lines of thrust lying wholly within the masonry; for the lower thrust line, Ungewitter has divided the flying buttress into seven segments, and constructed the corresponding funicular polygon. A similar technique will be used here for the detailed analysis of specific flying buttresses.

Figures 403, 404 and 405 show, in general terms, that the direction of thrust depends on the shape given by the builder to the flying buttress. The late (Flamboyant) buttress of Fig. 403 (St. Ouen, Rouen) was constructed without that firm intuitive grasp of structural principles shown in the twelfth and thirteenth centuries. Figure 406 (Amiens) is of exceptional interest, and is discussed in detail below.

The broad picture of the behaviour of flying buttresses indicated in Fig. 11 is certainly correct. A more detailed study, using the analytical techniques developed above, enables a more precise assessment to be made of the action of any particular flying buttress.

Consider first the inclined plate-bande of Fig. 7(b) as an idealized model of the flying buttress. Supposing the buttress to sag when the centering is struck, then three hinges will form at I_0, I_l and E_x, as shown in Fig. 12; the broken line shows the approximate

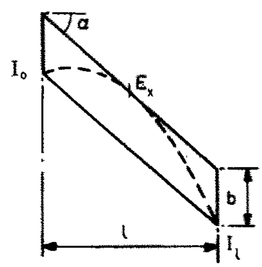

FIG. 12. Inclined plate-bande as model of flying buttress.

position of the line of thrust. At sections I_0 and I_l the line of thrust passes from the flying buttress into the nave vault and main buttress, respectively; the immediate problem is the location of the section E_x where the line of thrust just touches the extrados. There are several methods of attack; for interest, Coulomb's method will be used.

If the weight per unit (horizontal) length of the buttress is ω, then Fig. 13(a) shows the forces on the complete buttress, and Fig. 13(b) those on a portion of the buttress of length x. By moments about I_l in Fig. 13(a).

$$V + H \tan \alpha = \frac{\omega l}{2}$$

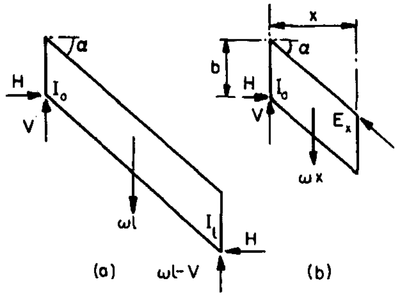

FIG. 13. Forces on inclined plate-bande.

and by moments about E_x in Fig. 13(b)

$$V+H\left(\tan\alpha-\frac{b}{x}\right)=\frac{\omega x}{2}$$

hence

$$H=\frac{\omega}{2b}x(l-x).$$

Now Coulomb states that the value of x (i.e. the position of E_x) must be such that the value of H is a maximum; hence $x = l/2$,

$$H=\frac{\omega l^2}{8b}$$

$$V=\frac{\omega l}{2}\left(1-\frac{l}{4b}\tan\alpha\right).$$

The same results would, of course, have been obtained by writing the condition that the line of thrust just touches the extrados at E_x.

The values of V and H determined above represent the *passive state* of the buttress; the buttress leans against the nave wall with this minimum value of $H = \omega l^2/8b$, and any shift inward of the nave wall (or outward of the main buttress; i.e. an increase in span of the flying buttress) will take place with V and H remaining constant.

Now this calculated value of H $(= \omega l^2/8b)$ must always be less than the horizontal component of the vault thrust; the latter must exceed the value $\omega l^2/8b$ if the nave wall is

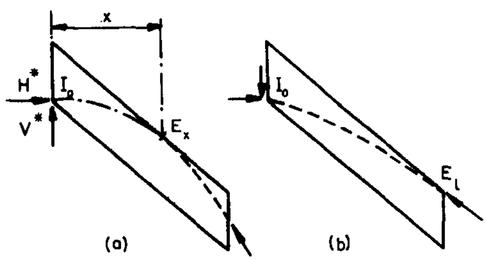

FIG. 14. Active thrust lines.

not to be pushed inwards. Consider, then, the same flying buttress, on which is imposed an increased horizontal thrust H^*, Fig. 14(a). It may be shown that, for the line of thrust sketched, $x^2 = 2bH^*/\omega$, and $V^* = \sqrt{(2b\omega H^*)} - H^* \tan\alpha$. This solution holds for $x < l$, i.e. $H^* < \omega l^2/2b$; for values of H^* greater than this, the line of thrust passes straight through the flying buttress, from nave to main buttress, as sketched in Fig. 14(b). It is clear that, since there is no possible mechanism of collapse for the plate-bande, the value of H^* can be increased until the crushing strength of the stone is reached.

Passive and active lines of thrust are reproduced in Fig. 15. It will be remembered that no sliding is allowed between individual stones, it being assumed that friction is sufficient to prevent this. Figure 15 shows that the passive line of thrust tends to "droop" at the head of the flying buttress, near I_0, and if the masonry is cut to resist the active thrust line (due, perhaps, to wind loading), then some sliding might occur at the head in the passive state.

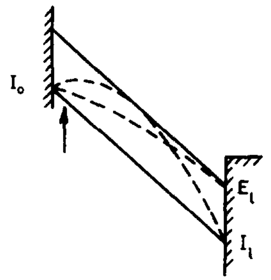

FIG. 15. Passive and active thrust lines.

An extra prop to the head of the flying buttress is often provided (see, for example, Fig. 16, St. Denis, Fig. 17, Beauvais, Fig. 18, Amiens, and Fig. 19, Clermont-Ferrand). Viollet-le-Duc attributes the failure of Beauvais to the fracture of the twin columns A (Fig. 17) leading to the fracture of the lintel L, and sliding failure at the head of the lower flying buttress (carrying the vault thrust).

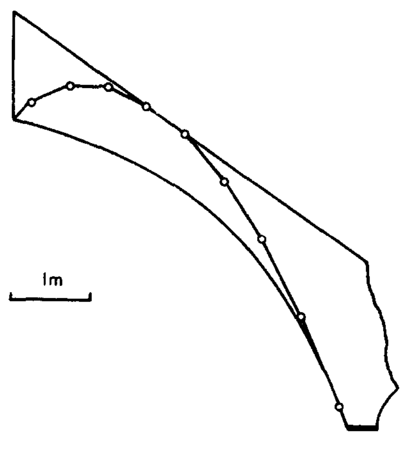

FIG. 20. Passive line of thrust, Clermont-Ferrand.

The upper flying buttress in Fig. 19 (Clermont-Ferrand) has been analysed in detail. The figure was traced, and the flying buttress divided into a number of sections, the weight of each section being concentrated at its centre of gravity. In this way the passive thrust line of Fig. 20 was calculated; the hinge point on the extrados was found by trial and error, using Coulomb's method.

From this passive thrust line, and assuming a weight of flying buttress of 10 ton, the upper buttress leans against the nave with a minimum horizontal thrust of 3 ton. It would be called upon to resist a maximum wind load of perhaps 20 ton (see Fitchen [19]).

Figures 21 and 22 give similar results for the great flying buttresses of Notre-Dame in Paris. The passive thrust in Notre-Dame is four to five times that of Clermont-Ferrand, since the span is so much greater; otherwise, Figs. 20 and 22 are very similar. In particular the droop in the thrust line at the head of the flying buttress is very marked.

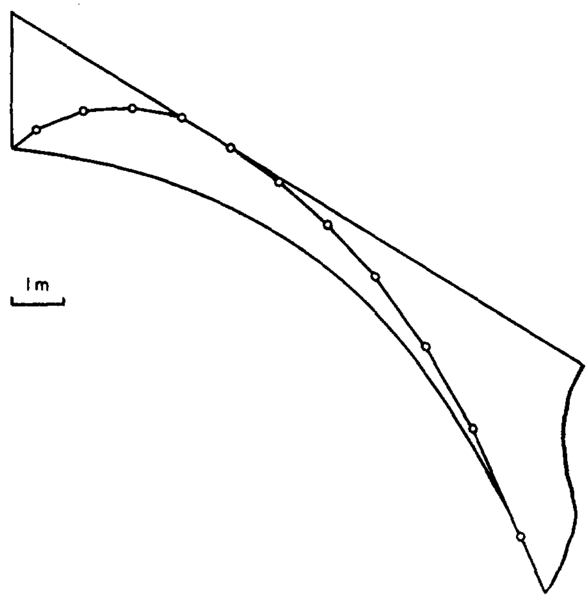

FIG. 22. Passive line of thrust, Notre-Dame.

From this point of view, the Gothic builders in France never quite mastered the flying buttress in the way that it was mastered, at about the same time, in England. Figures 23 and 24 give results for Lichfield Cathedral; it will be seen that the passive thrust at the head of the flying buttress is almost exactly horizontal. (Lichfield, and English Cathedrals in general, are lower and less slender than the French, and differences in the buttressing systems are to be expected.)

The computed figures for Lichfield may serve to illustrate the typical action of flying buttresses. The passive thrust is about 3 ton; the maximum active thrust, to cause crushing at the minimum cross-section of the buttress, is about 1000 ton. Taking a factor of $\frac{1}{10}$ on this last figure, it may be said that the flying buttress can work happily at a thrust anywhere between 3 and 100 ton, and that the flying buttress will adjust itself, automatically and exactly, to resist any value of thrust, live or dead, within that range.

Thus the flying buttress at Lichfield will actually be subject to an augmented passive thrust, exactly counter-balancing the dead value of the vault thrust. In addition the flying buttress at Lichfield will resist live wind loads. These dual functions of flying buttresses are sometimes split, and two separate buttresses are provided; see, for example, St. Denis, Fig. 16, Beauvais, Fig. 17, and Reims, Fig. 46. Figure 17 perhaps shows the arrangement most clearly; the lower buttress carries the dead vault thrust, while the upper (see Fitchen [19]) carries the wind load on the upper walls of the nave and on the great roof. In calm weather, the passive thrust in the upper flying buttress will have its minimum value, with the pronounced droop at the head.

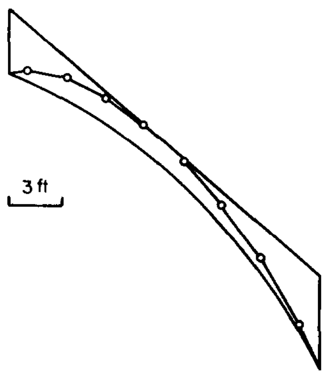

FIG. 24. Passive line of thrust, Lichfield.

As a final numerical example of the action of flying buttresses, those at Amiens, shown to an enlarged scale in Fig. 25, will be analysed. This illustration shows the buttresses at the chevet, as they were built, and as they still exist. Similar buttresses were provided originally to the nave, but these failed by upward buckling, and were replaced by a different design in the fifteenth century.

The flying buttresses to the nave are subject to much higher wind forces than those to the chevet. It will be seen from Fig. 25 that wind loads can be taken by the straight upper rib of the flying buttress, which is separated by tracery from the curved lower rib. If, however, the curved lower rib itself buckles, then the straight upper rib will be pushed aside and will be useless as a wind brace. This appears to have been the mode of failure.

A numerical analysis may be made for the lower rib, taking into account the weight of the upper rib transmitted through the short tracery columns. Figure 26 shows, as usual, the passive line of thrust; this corresponds to a passive horizontal thrust of about 5 ton. The other line of thrust in the figure corresponds to the maximum active horizontal thrust if the flying buttress is to remain stable. Instead of an effectively infinite value of this thrust, the curvature of the lower rib at Amiens limits the value to almost exactly four times the passive value; corresponding to the figure of 5 ton, the maximum value of horizontal thrust is therefore 20 ton, of the same order as the maximum wind load.

At this maximum value of thrust, the lower rib starts to buckle upwards, Fig. 27, pushing the inclined upper rib aside, and allowing the nave wall to lean outwards.

Figure 17 (Beauvais) and Figs. 18 and 25 (Amiens) show intermediate piers between the nave and the main external buttress. At Beauvais, for example, the double flying

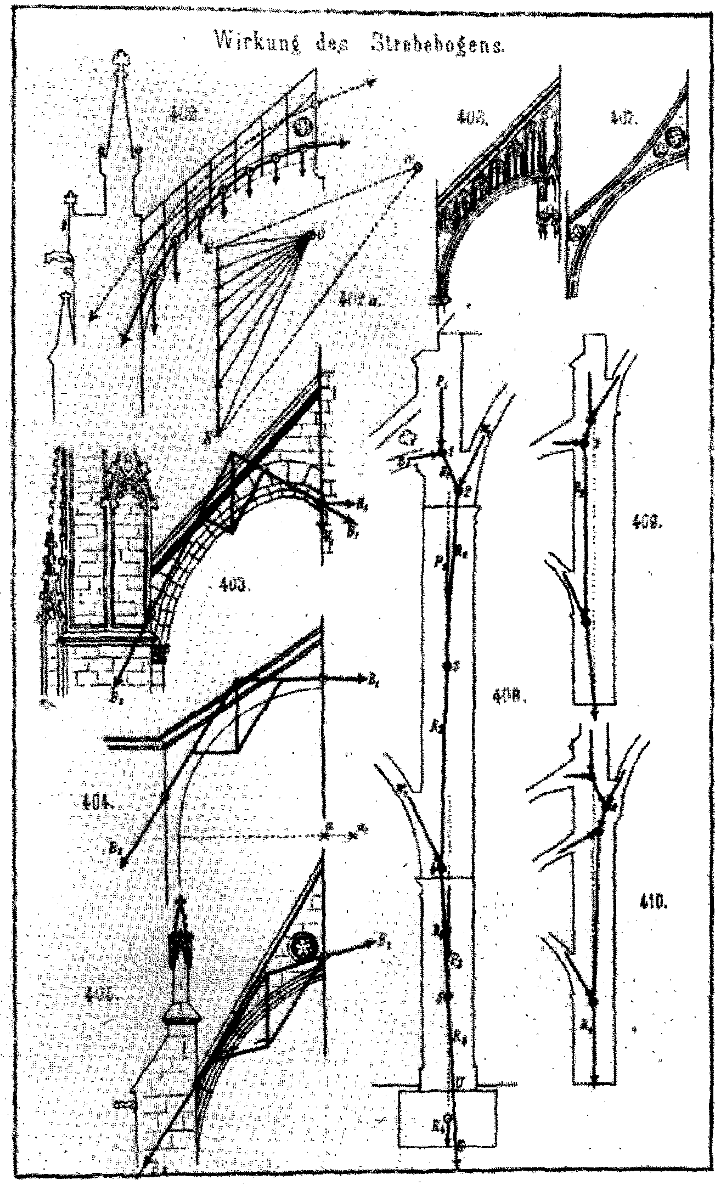

FIG. 11. Ungewitter's Plate 41; flying buttresses.

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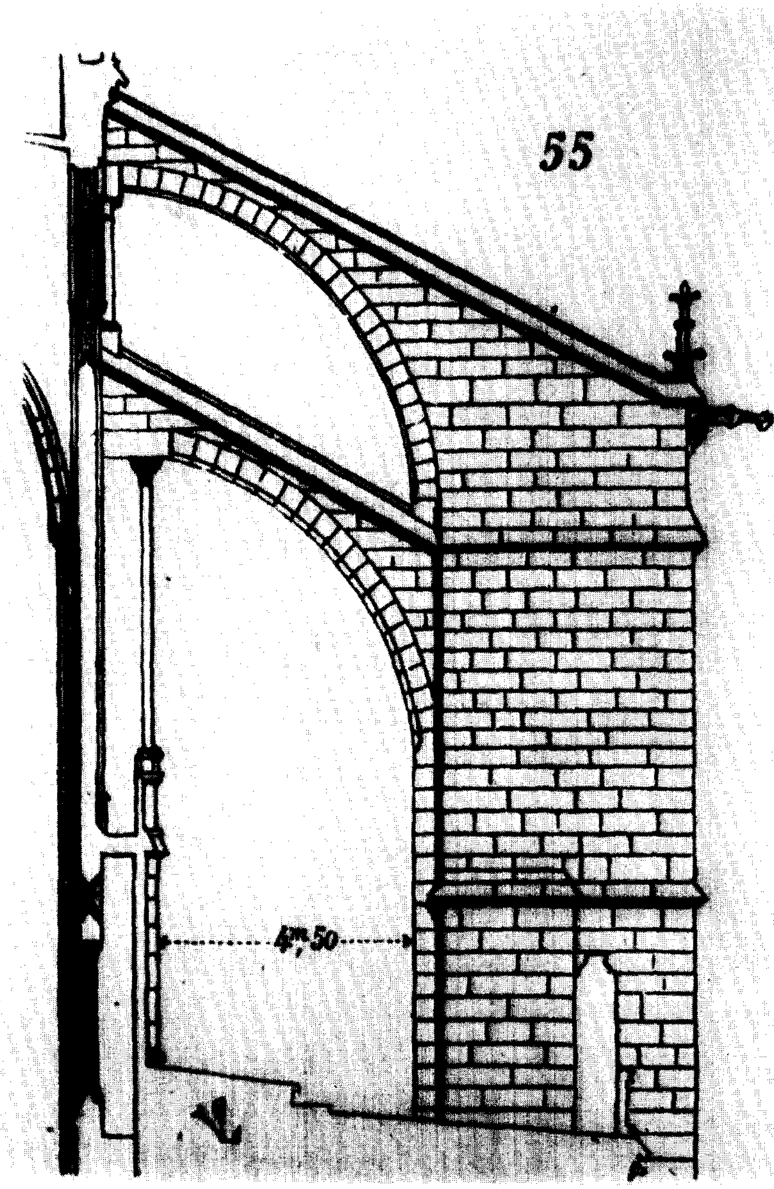

Fig. 16. Viollet-le-Duc's drawing of St. Denis.

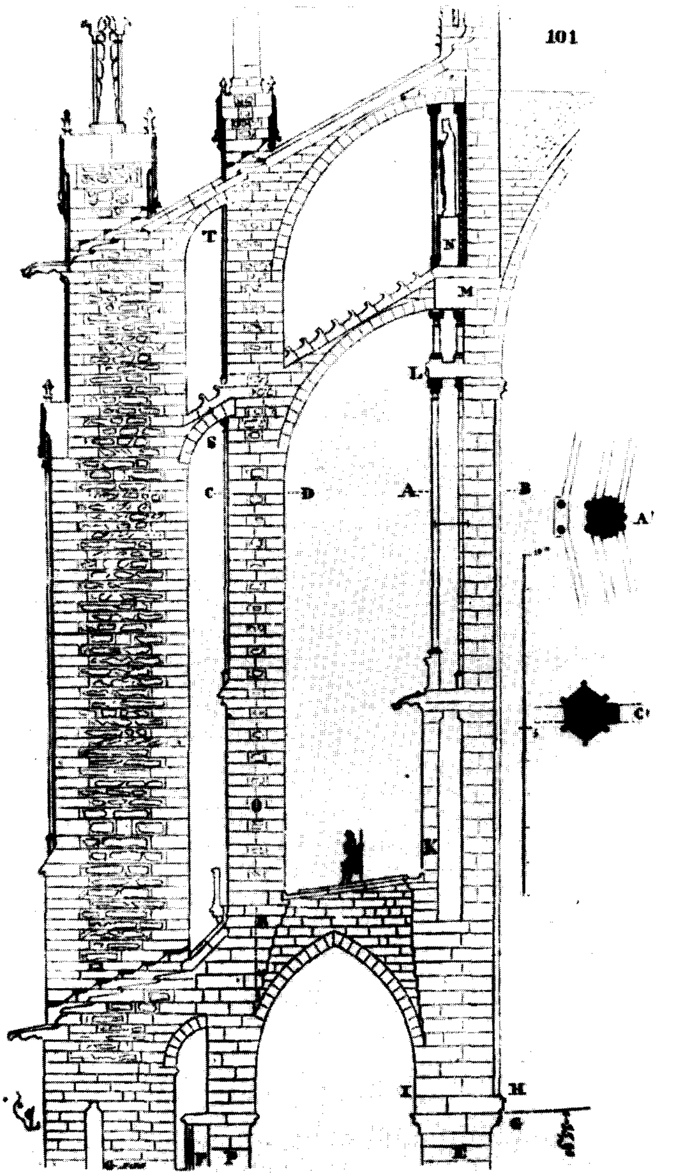

Fig. 17. Viollet-le-Duc's drawing of Beauvais.

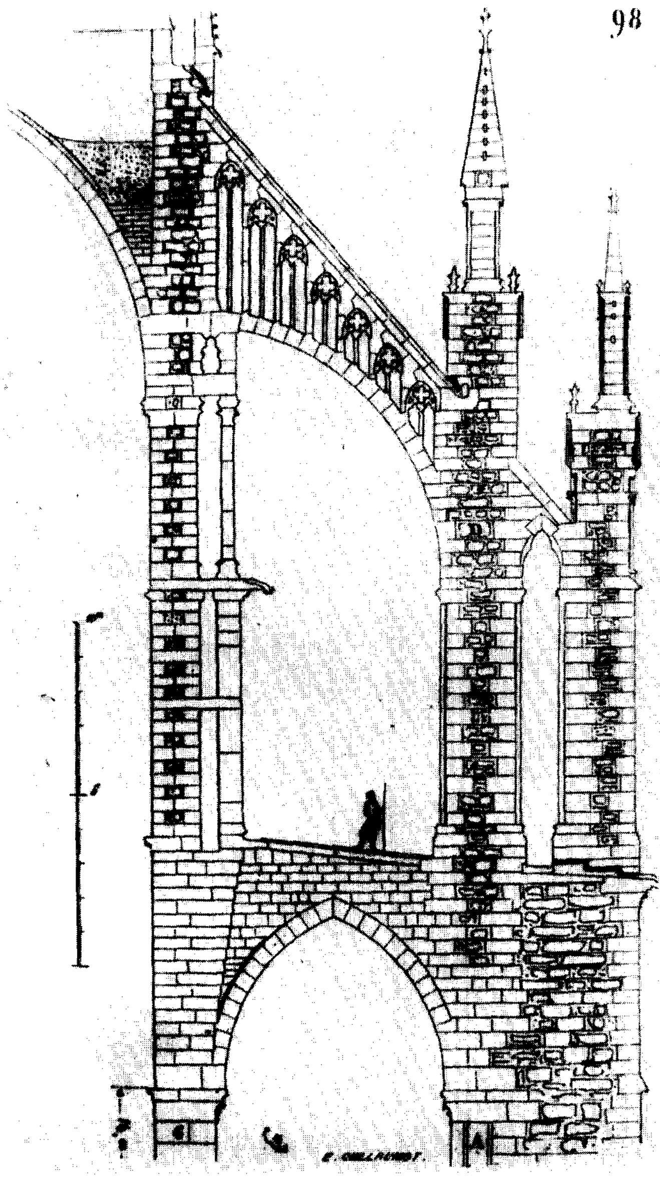

FIG. 18. Viollet-le-Duc's drawing of Amiens.

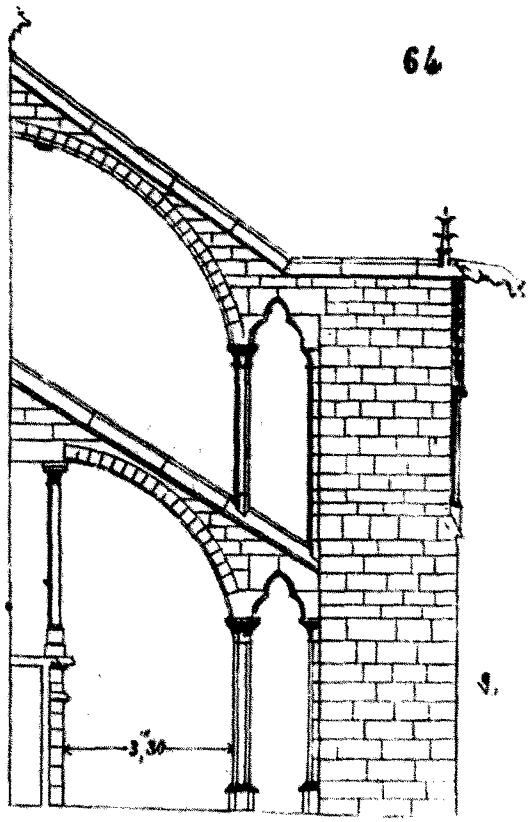

FIG. 19. Viollet-le-Duc's drawing of Clermont-Ferrand.

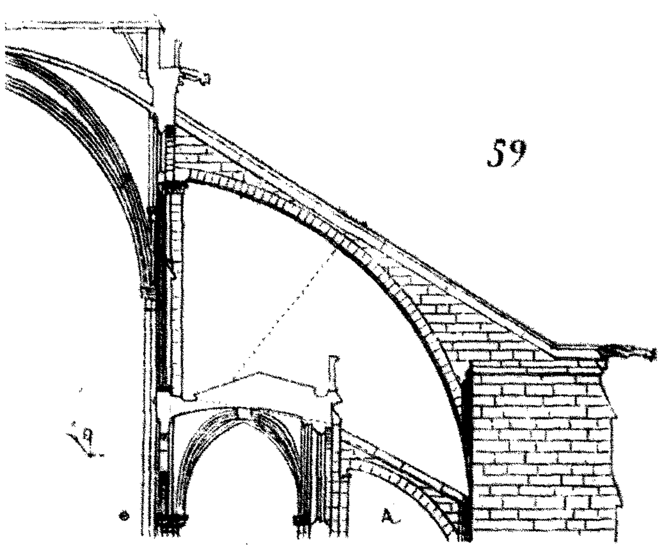

FIG. 21. Viollet-le-Duc's drawing of Notre Dame, Paris.

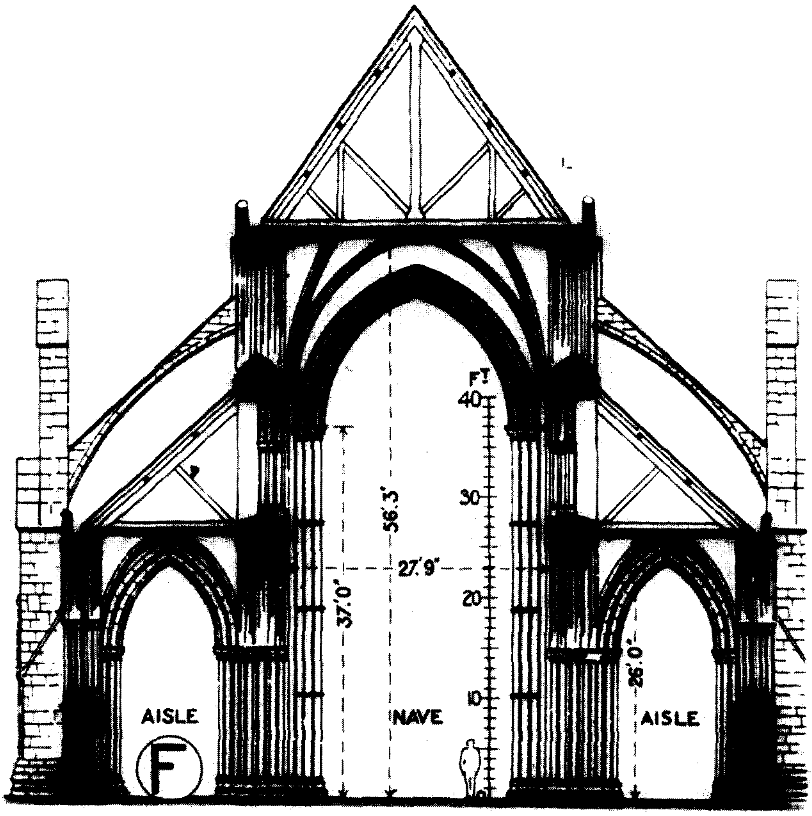

FIG. 23. Sir Banister Fletcher's drawing of Lichfield [39].

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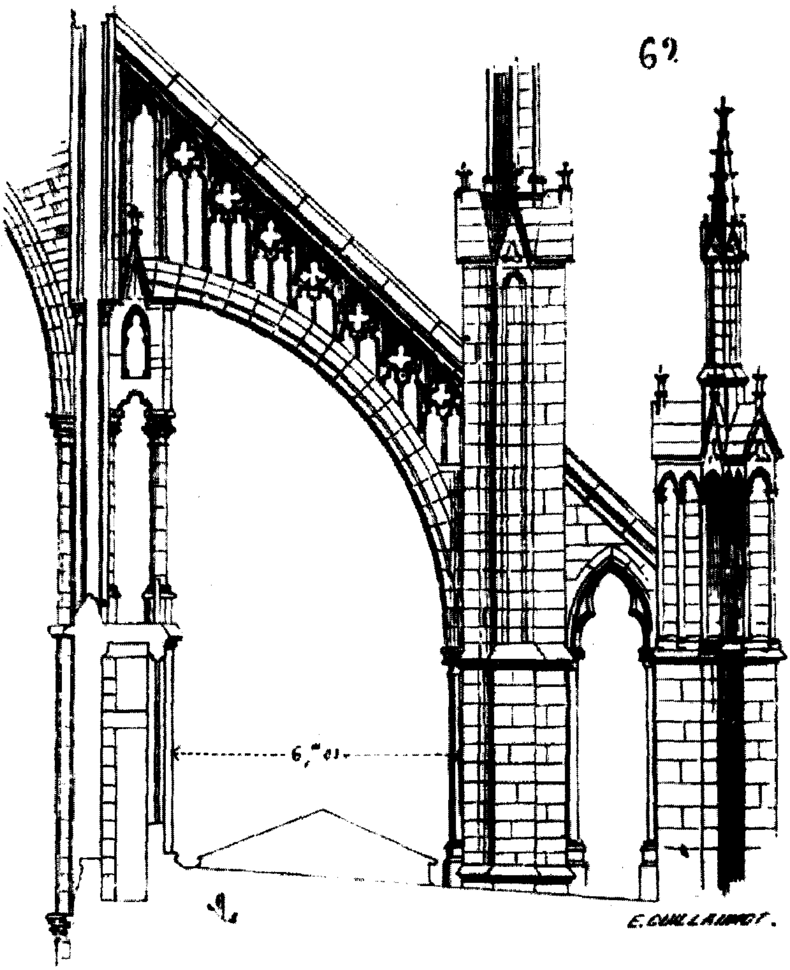

FIG. 25. Viollet-le-Duc's drawing of Amiens.

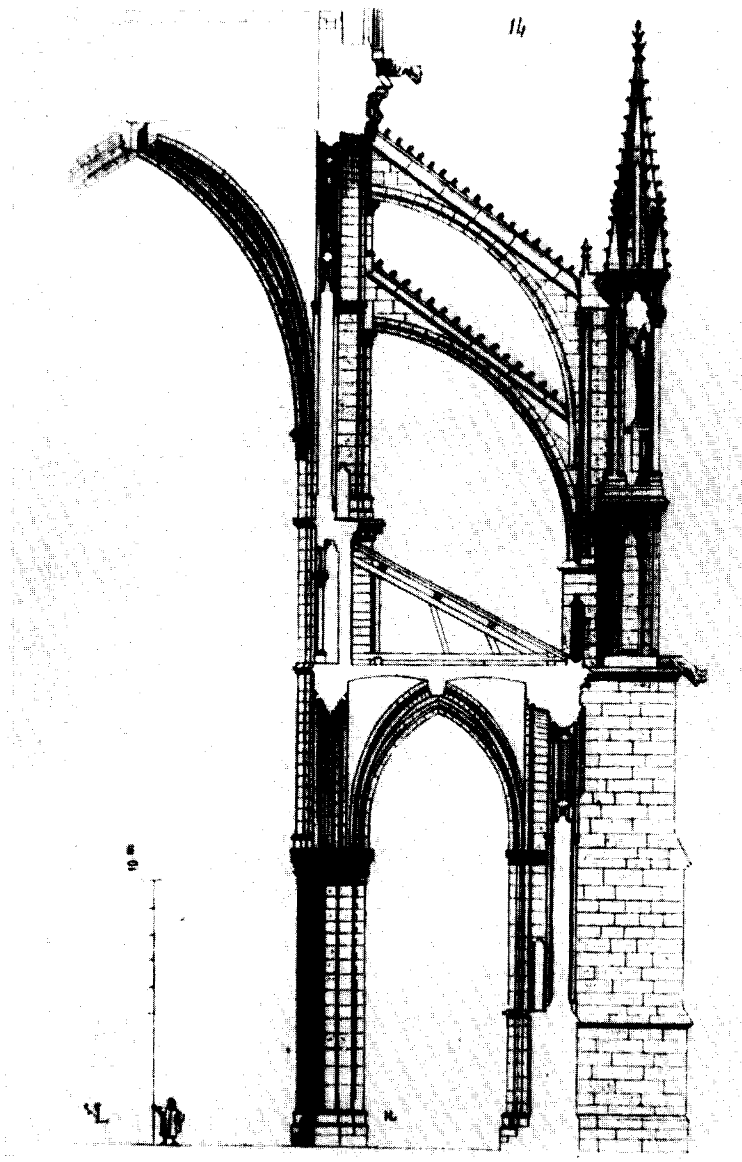

FIG. 46. Viollet-le-Duc's drawing of Reims.

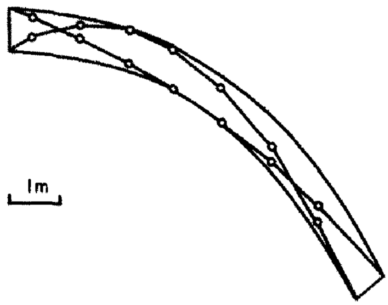

FIG. 26. Passive and active lines of thrust, Amiens.

buttress system terminates in the first instance at the intermediate pier. Small flying buttresses (S and T, Fig. 17) are used in turn to prop the intermediate pier from the main buttress. For such large overall spans, the only alternative would be to use very large flying buttresses like those of Notre-Dame (Fig. 21). The passive thrusts from such large flying buttresses are, of course, much higher than those developed by a structural system using an intermediate pier.

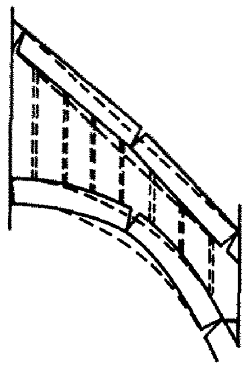

FIG. 27. Possible mode of failure, Amiens.

VAULTS AND DOMES

As mentioned earlier, domes disperse forces. It was the aim of the Gothic builders to provide a stone covering to the nave (for fireproofing) while at the same time allowing the structure to admit as much light as possible. Thus the nave vault was required to concentrate forces, so that the intermediate structure could be cut away. Inevitably, therefore, the vaulting shell would be very highly stressed in the regions where it connected to the main structure, since it is only at those points that load could be transmitted. It is the unique achievement of the Gothic builders that they realized that a smooth shell would be overstressed, and that the only practicable solution involved the assembly of portions of shells upon structural ribs. By this means the stress level in the shell proper can be kept low and uniform, while the diagonal ribs in the vaulting compartment take the load back to the main piers.

Thus, when Viollet-le-Duc (article "Cathédrale") writes: "Already in 1220 Guillaume le Breton talks of its (Chartres) vaults 'which', he says, 'can be compared to the shell of a tortoise' ", the wrong impression is given, because the shell of a tortoise is smooth, and not creased. The groin creases, in themselves a weakness for the shell of a Gothic vault, are reinforced by masonry ribs to produce a stiff and light structural unit of great inherent strength.

Fitchen remarks "... nothing has so far been written regarding the application of the thin-shell theory to the webs of Gothic ribbed vault construction". Such an application will be made here, from which one or two interesting points will emerge. However, the major fault forces may be determined almost completely by simple statical considerations.

Figure 28, redrawn from Fitchen, shows two bays of a quadripartite vaulting system, in which each bay is roughly a 2 by 1 rectangle. The groins are stiffened by ribs, and these would have been erected first on timber centering. With the main transverse ribs, the all-important diagonal ribs, and the window formerets completed, the vault shell would have been laid in courses in thick mortar. With one bay completed, the centering to the ribs would have been struck, and re-erected in the next bay.

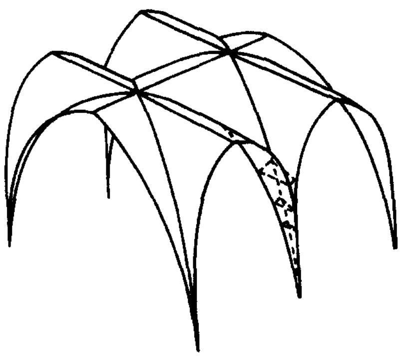

FIG. 28. Quadripartite vaulting system (after Fitchen).

The stresses in the shell, and the diagonal ribs, will be investigated, and an answer attempted to a particular and curious problem. The vaulting conoid, illustrated by the triangular and trapezoidal sections in Fig. 28, is almost invariably filled with rubble masonry up to over half the height of the vault, i.e. up to about the level shown. Fitchen comments on the stabilizing action of this rubble fill, which can be seen clearly in the cross-sections of Beauvais, Fig. 17, and Amiens, Fig. 18. Mackenzie [30] shows similar fill in his cross-section of King's College Chapel, as does Willis [31] in his cross-sections of Peterborough Cathedral and St. George's Chapel, Windsor.

A start may be made on this problem by examining the behaviour of domes, rather than vaults. The well-known membrane solution (see, for example, Flügge [32]) for a hemispherical dome of uniform thickness consists of two principal stress resultant components (N_φ, N_θ). In Fig. 29,

$$N_\varphi = -\frac{\omega a}{1+\cos\varphi}$$

and the stress resultant N_θ in the hoop direction is given by

$$N_\theta + N_\varphi = -\omega a \cos \varphi$$

In these expressions, ω is the weight per unit area of the shell.

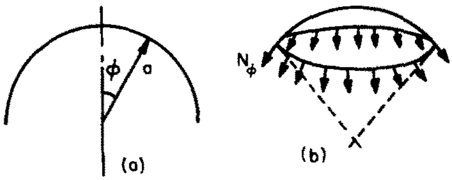

FIG. 29. Hemispherical dome.

For small φ, near the crown of the dome, both stress resultants are compressive. As φ increases, however, the hoop stress N_θ changes sign and becomes tensile; this occurs at $\varphi \sim 52°$. A masonry dome will therefore tend to crack near its base; for this reason, Fontana and della Porta introduced the iron rings when building the dome of St. Peter's, Rome, at the end of the sixteenth century (see, for example, Straub [10]).

As is well known, the dome nevertheless cracked badly, and was the subject of reports by, among others, Poleni [33]. In his analysis, Poleni observed that an arch subjected to uniform loading should be proportioned to have the shape of an inverted catenary, and he used the same ideas for the dome. Considering thin "orange slices" of the dome (Fig. 30), each separate from its neighbours, he deduced the line of thrust for this non-uniformly loaded element, and showed that this line of thrust in fact lay within the masonry of the dome as built.

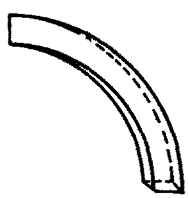

FIG. 30. Slice of hemispherical dome.

He concluded that the cracks were due to unequal settlements, and so on, rather than potential instability, and were therefore not dangerous; he agreed with other engineers that additional iron bands should be provided.

Poleni's approach is completely justified from the viewpoint of limit design; he obtained a statically admissible solution which was therefore safe. In formal terms, a thrust line was obtained, not for the original dome, but for a dome cut into segments. The actual solution for the "sliced" dome is clearly statically admissible for the original dome. It will be recalled that, in effect, Yvon Villarceau used the same technique for arches.

Gregory [34] has some interesting remarks on the philosophy of this sort of statical approach to structural design. Siegel [35] quotes the case of the Frauenkirche, Dresden (destroyed in World War II) which was the subject of an investigation in the nineteen thirties. Similar cracking had been observed, and it was demonstrated that the dome was acting not as a shell but as a series of "orange-slice" arches. Again the dome was safe, and it is of interest to determine how thin a dome can be made if it is to remain stable.

Siegel gives the following table:

	Span, L (m)	Thickness of shell, d (cm)	L/d
St. Peter's, Rome	40	300	13
Frauenkirche, Dresden	24	125	19
Hen's egg	0·04	0·04	100
Zeiss Planetarium, Jena	40	6	667
Central market, Basle	60	8·5	700
Exhibition hall, Paris	205	13	1570

The very thin shells can only be built, of course, (as pointed out by Goethals [36], for example), by using *reinforced* concrete, i.e. by using steel to absorb tensions which are developed.

The lower L/d ratios can be (and were) achieved without reinforcement, and Poleni's analysis may be applied to the hemispherical dome to give a safe estimate of the minimum thickness. By considering a slice of the type sketched in Fig. 30, the line of thrust may be determined. This line of thrust departs from a circle by just over ± 2 per cent of the mean radius. Thus, if a hemispherical dome were proportioned to have thickness about $4\frac{1}{2}$ per cent of the radius ($L/d = 45$), a statically admissible line of thrust could be found, based on the consideration of the weaker, sliced dome. This dome is drawn to scale in Fig. 31, and the line of thrust is also shown, just contained within the masonry. (As a safety factor against instability, a practical shell should be thickened slightly, to ensure that the thrust line remains within the masonry despite accidental movements of the structure.)

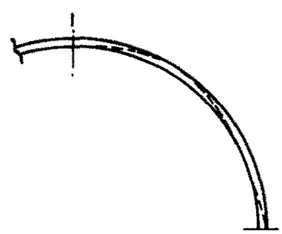

FIG. 31. Statically admissible thrust line for hemispherical dome.

The conclusion is that a hemispherical dome of uniform thickness $4\frac{1}{2}$ per cent of the radius may be built from masonry without reinforcement, and that that dome will be stable. Indeed the $4\frac{1}{2}$ per cent is an overestimate, and a thinner dome would stand.

It is important to note the differences between the safe statically admissible solution of Poleni and the membrane solution given earlier. In the membrane solution, the shell thickness is supposed to be infinitesimal, and all forces act in the surface of the shell.

The "orange-slice" solution, however, allows the line of thrust to depart from the centre line of the shell; referred to that centre line, the thrust is accompanied by bending. In the example above, however, if the dome thickness is $4\frac{1}{2}$ per cent of the radius, the bending will be absorbed by purely compressive stresses.

(As a matter of interest, it is possible to obtain a membrane design of dome involving only compressive stresses, analagous to the inverted catenary in two dimensions; profiles of such domes have been worked out by, for example, Bouteloup [37]).

The membrane solution involves tension in the material at the base of the hemispherical dome; the "orange-slice" solution involves a line of thrust which is inclined at the springing, and which therefore also requires an inward containing force, perhaps provided by an iron hoop. In either case, trouble is encountered near the base of the dome, where external reinforcement is required. To avoid this trouble, the dome must be cut off, so that it is no longer a full hemisphere, or thickened near the base and merged into another part of the structure so that it is no longer, in that region, a thin shell.

It will be seen that the rubble fill in the conoid of a Gothic vault fulfils an analogous purpose. It allows propping forces to be applied to the vault shell, while at the same time allowing the thin shell to merge into the solid masonry fill. Breymann knew of this function in two dimensions; his Fig. 11 (in Fig. 5 here) shows such a fill between semi-circular arches.

THE RIB VAULT

As a first approximation to the Gothic vault, consider the intersection of two semi-circular barrels of radius a, shown in perspective in Fig. 32(a) and plan in (b). Taking axes

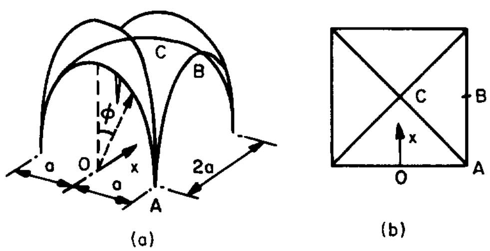

FIG. 32. Semi-circular rib vault, square compartment.

as shown, and following Flügge, the *membrane* forces on an element will be as shown in Fig. 33, where ω is the weight of the shell per unit area. By resolving forces in the radial, hoop, and axial directions,

$$N_\varphi = -\omega a \cos\varphi$$

$$\frac{\partial N_{x\varphi}}{\partial x} = -\omega \sin\varphi - \frac{1}{a}\frac{\partial N_\varphi}{\partial \varphi}$$

$$\frac{\partial N_x}{\partial x} = -\frac{1}{a}\frac{\partial N_{x\varphi}}{\partial \varphi}$$

and these equations may be solved to give

$$N_{\varphi} = -\omega a \cos \varphi$$
$$N_{x\varphi} = -2\omega x \sin \varphi + f_1(\varphi)$$
$$N_x = \frac{\omega x^2}{a} \cos \varphi - \frac{x}{a} \frac{df_1}{d\varphi} + f_2(\varphi)$$

where the arbitrary functions of integration $f_1(\varphi)$ and $f_2(\varphi)$ must be determined from the boundary conditions of the problem.

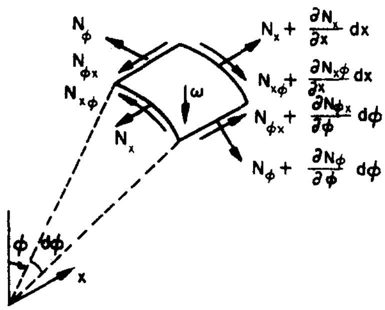

FIG. 33. Element of cylindrical shell.

It should be noted that the stress resultants have been obtained directly from the equations of equilibrium, without using any compatibility statements or the stress–strain law. The membrane solution is, in fact, statically determinate, and depends only on the unit weight and the dimensions of the shell.

It may be observed first that the hoop stress resultant N_{φ} is uniquely determined without reference to the boundary conditions (and is, in fact, dependent only on the local curvature of the shell); at $\varphi = 0$ it has the value $-\omega a$. For the particular problem of Fig. 32, the function f_1 is zero everywhere by symmetry. If then a transverse cut be made through the vault as in Fig. 34, the only shell forces acting at the cut will be the uniform hoop compression of value ωa/unit length.

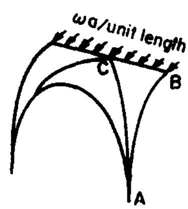

FIG. 34. Vault cut at crown, perspective.

The side elevation of the cut vault is shown in Fig. 35, and the horizontal force of total magnitude $2\,\omega a^2$ can be seen acting at the level of B. Neglecting the weight of the ribs, the half-vault has weight $W = \omega a^2(2\pi - 4) = 2{\cdot}28\,\omega a^2$, acting at a distance $(9\pi/8 - 3)a/(\pi - 2)$ $= 0{\cdot}468a$ from the free edge. This vertical force must be resisted by an equal vertical force W acting at A, and for complete equilibrium, a horizontal force $2\omega a^2$ must act at a height h below the top of the vault, where $(2\omega a^2)(h) = W(0{\cdot}468a)$, i.e. $h = (9\pi/8-3)a = 0{\cdot}534a$.

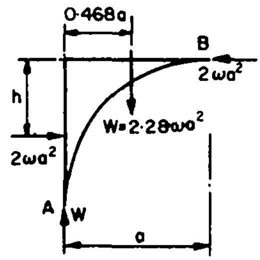

FIG. 35. Vault cut at crown, elevation.

From where does this horizontal force arise? If the vault of Fig. 32 represents one bay of a complete nave vault, of which two bays are shown in Fig. 36, it will be seen that there is no problem in the longitudinal direction; each shell will "lean" against the next with the required force. (There will, of course, be erectional problems. The completion of a compartment of vaulting will buttress all previously completed compartments, but the centering must be arranged to stabilize, in the *horizontal* direction, the final exposed transverse rib.)

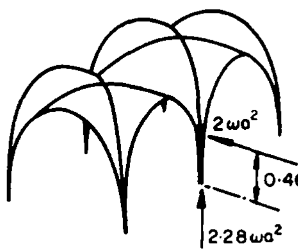

FIG. 36. Vault thrust supplied by flying buttress.

However, an external force must be provided, as shown in Fig. 36, in the transverse direction. This force is the *vault thrust*, to be counteracted by the flying buttress; ideally, the head of the flying buttress should lie about halfway between the springing and the crown of the vault. (Note the positions of the lower buttress of St. Denis, Fig. 16, of the tas-de-charge M of Beauvais, Fig. 17, and of Amiens, Fig. 18, all three appearing to be low, and of the well placed great buttresses of Notre-Dame, Fig. 21, Lichfield, Fig. 23 and Reims, Fig. 46.)

Now if the flying buttress applies a thrust at the level indicated in Fig. 36, the vaulting conoid must be made solid in order that the thrust can be transmitted to the shell; thus the conoid must be filled, for this purpose, at least to the level of the line of action of the vault thrust.

Assuming that overall equilibrium is satisfied as in Fig. 35, it is possible to determine next the forces in the diagonal ribs along the groins, which are, again, a function of the value of N_φ and of the weight of the shell. Each rib force F has value

$$F = \frac{2\,\omega a^2\sqrt{(2+\tan^2\varphi)}}{\tan\varphi}[\sin^2\varphi\cos\varphi-\sin\varphi\cos\varphi+\varphi+\cos\varphi-1]$$

and this is plotted in Fig. 37. There are no forces in the transverse ribs or the formerets, other than those due to their own weight.

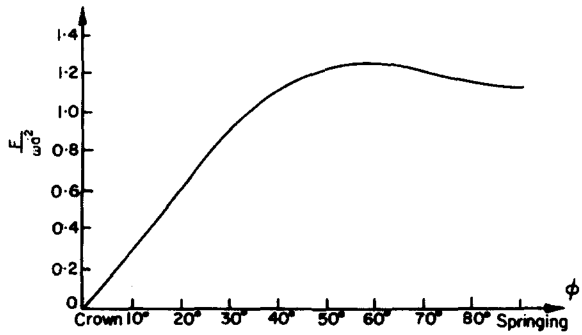

FIG. 37. Force in diagonal ribs.

The diagonal rib forces F, however, are large, and the equilibrium solution derived so far, and which describes very accurately the mode of action of the vault, is possible only if the shell is *reinforced* along the groins by the diagonal ribs; the "creases" in a shell are lines of potential weakness, and must be strengthened in this way. Thus the diagonal vault ribs serve (a) as a necessary structural element of the completed vault, (b) as permanent formwork for erecting the shell, and (c) as a mask to hide the joint between two shell surfaces.

Proceeding with the analysis, the three stress resultants become

$$N_\varphi = -\omega a\cos\varphi$$

$$N_{x\varphi} = -2\omega x\sin\varphi$$

$$N_x = \frac{\omega x^2}{a}\cos\varphi + f_2(\varphi)$$

The function $f_2(\varphi)$ represents the axial thrust (vault thrust) distributed round the semi-circular shell at the section $x = 0$, and has a total integrated value $2\omega a^2$; as has been seen, this thrust is applied either by the adjacent portion of the next vaulting bay, in the longitudinal direction, or by the flying buttresses, in the transverse direction. The function $f_2(\varphi)$ is closely related to the rib thrusts F; the expression is lengthy, and is shown graphically for a quarter circle in Fig. 38.

This completes the exact membrane solution for the square right circular cylindrical vault of Fig. 32. A modern use of such vaults may be found at the Air Terminal, St. Louis (see Becker [38]). Here the three vaulting compartments are each 120 ft square, the reinforced concrete shells are $4\frac{1}{2}$ in. thick, and the diagonal ribs are cast externally. The span to thickness ratio is thus 320; the thickness to radius ratio is about 0·5 per cent. The vaults are not complete semi-circles, but are cut off at about 55° from the vertical, that is, at almost exactly the section at which the vault thrust should be counteracted.

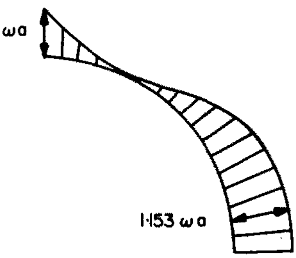

FIG. 38. Vault thrust distributed round vault shell.

To move a little towards the solution for an actual vault, consider first the same square bay of Fig. 32, but with the flying buttresses wrongly placed. Suppose, for example, that the tas-de-charge is placed exactly at the springing of the vault, so that the horizontal thrust is applied as in Fig. 39 (cf. Fig. 35). The weight W of the vault (assumed equal to $2{\cdot}28\,\omega a^2$), and the line of action of this weight, are of course fixed; in Fig. 39, there must be a clockwise couple of magnitude $(2{\cdot}28)(0{\cdot}468)\,\omega a^3 = 1{\cdot}07\,\omega a^3$, and this must be resisted by an equal anti-clockwise couple. Thus in Fig. 39 the horizontal forces have magnitude $1{\cdot}07\,\omega a^2$, and this value is incompatible with the local value of N_φ at the crown, which led to the figure $2\,\omega a^2$ in Fig. 35.

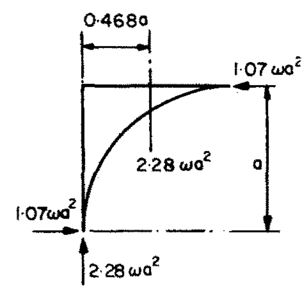

FIG. 39. Vault thrust reacted at the springing.

Now N_φ depends on the local curvature of the shell; if the vault is to be stable, it must deform at the crown (to approximately double the curvature) to give the total force $1{\cdot}07\,\omega a^2$ of Fig. 39. More precisely, since an infinitesimally thin shell has been assumed, while the actual vault has a finite thickness, the line of thrust in the vault, while staying within the masonry, must depart from the centre line of the shell. It will be seen below that the use of a pointed vault leads to a reduction in vault thrust, and thus allows a lower flying buttress than is required for the cylindrical vault.

There is, moreover, a second consequence of a low flying buttress, which can perhaps best be seen from Fig. 36. If the horizontal force acts at the springing level, it is clear that this force will cause severe (and inadmissible) bending in the slender vaulting conoid, unless that conoid is again filled with masonry to transmit the force in compression. The rubble fill will, itself, help to compensate statically for a low flying buttress; in Fig. 39 the rubble fill will contribute an extra (if small) clockwise moment about the springing, thereby permitting an increase in the vault thrust above the value $1{\cdot}07\,\omega a^2$.

An actual vault in a cathedral is (a) over rectangular rather than square compartments, (b) not circular but pointed, (c) not cylindrical, but commonly slightly domed, and (d) often built so "badly" that on close inspection it is seen to depart grossly from any recognizable mathematical surface. The membrane stress resultants obtained above for the cylindrical barrel vault are highly sensitive to such variations, and hence no meaningful conclusions can be drawn from any further analysis of this type. However, the general considerations of overall equilibrium must apply whatever the precise shape of the vault. If a cut be imagined to be made down the centre line of the nave, then figures similar to Figs. 35 or 39 may be constructed, and vault thrusts determined to a high degree of accuracy, almost by inspection (an example is given below).

For investigating the stability of the shell itself, the technique of constructing safe statically admissible solutions may be used. The shell can be sliced, perpendicular to the axis, into a series of rings, each ring spanning between two of the diagonal ribs. It will be seen from Fig. 32 that near the crown C of the vault, the rings will subtend only a small angle at the centre, but that this angle (2α in Fig. 40) will increase up to 180° as the ring is taken closer to the edge B.

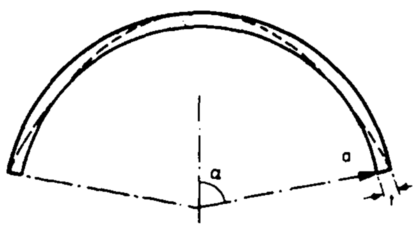

FIG. 40. Sliced ring of vault.

The ring in Fig. 40 has been assigned a thickness t just sufficient to contain the line of thrust, exactly as was done for the dome (cf. Fig. 31). Figure 41 shows this required thickness t as a function of the "cut-off" angle (the corresponding thicknesses are also given for the dome). It will be seen that the shell can be very thin near the crown, where the angle α is small, but should increase progressively away from the centre of the vault.

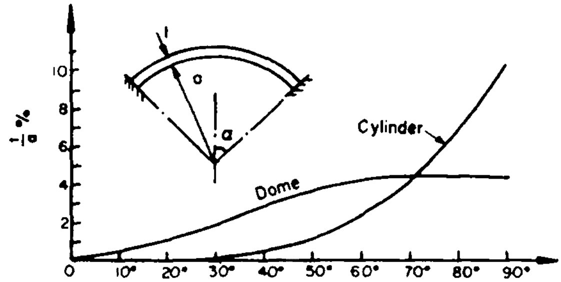

FIG. 41. Required thicknesses of cylindrical vault and of dome; safe solutions.

Figure 42 gives the shell thrust per unit length, N_φ, in terms of the cut-off angle. It will be seen that the thrust falls away slowly from the value ωa given by the membrane solution.

The results of Figs. 41 and 42 have been obtained on the (safe) assumption that there is no interaction between adjacent rings; a statically admissible solution has been obtained for the rings which must, *a fortiori*, be statically admissible for the original shell. Thus adjacent rings may have slightly different curvatures with a negligible effect on the analysis, and this in turn implies that a "wavy", badly built, vault would suffer no penalty in this statically admissible investigation of its stability.

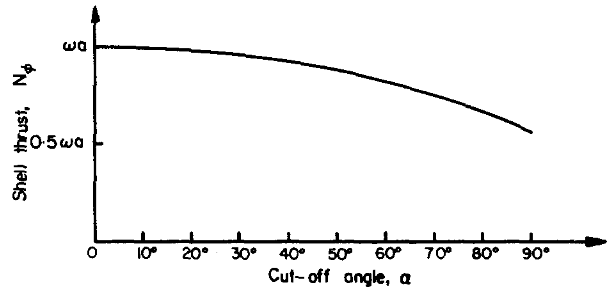

FIG. 42. Shell thrusts for sliced vault.

The analysis may be repeated for a pointed vault, for example that shown in Fig. 43, where the centre of the circle is displaced by an angle α_0. Considering again sliced arches, this time of pointed shape, the thickness of shell required to contain the thrust line may

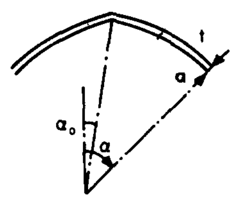

FIG. 43. Pointed arch.

be determined for any displacement angle α_0 and cut-off angle α. Figure 44 gives the required percentage thickness of shell for $\alpha_0 = 10°$ (i.e. a 160° "point"), and Fig. 45 the corresponding value of thrust per unit length at the crown. (The calculations have not been taken beyond $\alpha = 70°$.)

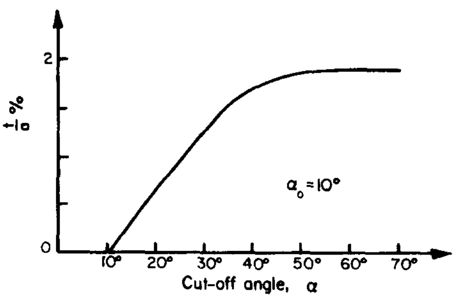

FIG. 44. Required thickness of pointed shell.

It will be seen that a vault having thickness 2 per cent of the radius may be used up to $\alpha = 70°$. The thrust at the crown increases rapidly from zero at the centre of the vaulting bay to something over $\frac{1}{2}\,\omega a$ per unit length.

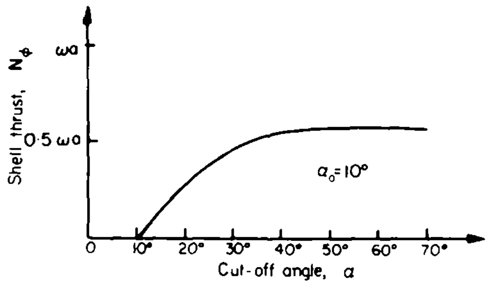

FIG. 45. Shell thrusts, pointed vault.

SOME ANALOGUES OF REIMS

Figure 46 is one of Viollet-le-Duc's cross-sections of Reims. The masonry is exceptionally heavy in this cathedral, possibly because the original design of Robert de Coucy was cut down after building had started; the foundations and nave piers imply a cathedral of colossal dimensions. Even so, the vaults are exceptionally thick, being nearly 24 in.; those of Notre-Dame are about 6 in., and a figure of 9 or 10 in. was very usual at about that period. (According to Mackenzie, King's College Chapel varies from 2 to 6 in.)

It is of interest to redesign various elements of the nave of a cathedral having the leading dimensions shown in Fig. 47; these dimensions are slightly larger than those of Reims.

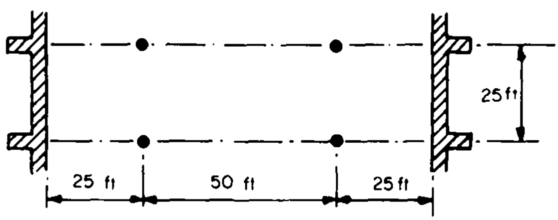

FIG. 47. Nave bay; approximate dimensions of Reims.

The main nave vault covers a rectangular bay 50 ft by 25 ft. Assuming a pointed vault of 160°, to which Figs. 44 and 45 apply, and assuming also a rubble fill in the conoid up to half the total height, so that the cut-off angle for the shell is about 60°, it will be seen from Fig. 44 that a 2 per cent shell will be satisfactory. Taking the radius, a, to be about 35 ft, the shell thickness required is therefore about 8 in.

Thus for stone at 150 lb/ft^3, the unit weight ω is 100 lb/ft^2. From Fig. 45 the thrust at the crown is therefore about $(\frac{1}{2})(100)(35) = 1750$ lb/ft (corresponding to the very low stress of about 18 lb/in^2); the total thrust across one bay (vault thrust) is thus $(1750)(25)/(2240) = 20$ ton, say.

Allowing a factor of 1·5 for the ribs, and for the curvature of the shell, the weight of a half-bay of vaulting is approximately $(1{\cdot}5)(100)(25)^2/(2240) = 40$ ton, say. Thus, referring to Fig. 48, where these values are entered, the thrust from the flying buttress (20 ton) should be applied about 20 ft below the crown of the vault, in almost exactly the position of the existing lower buttress (Fig. 46).

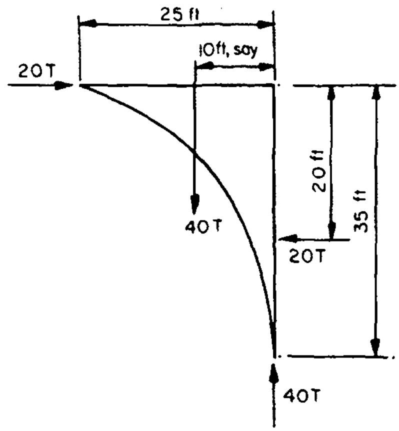

FIG. 48. Vault thrust, Reims analogue.

The maximum force in a diagonal rib will be about half the weight of half-bay of vaulting, i.e. about 20 ton. At 600 lb/in^2 a load of 20 ton requires a rib area of 75 in^2, say 10 in. × 8 in.

It is of interest to record that Fitchen [19] estimates the horizontal wind load to be taken by the upper flying buttress as about 15 ton, due to a unit wind pressure of 28 lb/ft^2 acting on the great roof at Reims. (The masonry bays at Reims are 23 ft, and not 25 ft as in Fig. 47.)

Fitchen estimates also the weight of the great roof itself at about 52 ton per masonry bay. Thus at each bay, there might be a total weight of 100 ton, acting above the upper flying buttress, due to the roof and to the massive stonework parapet. There is, in any case, plenty of dead weight available to counteract the inclined resisting thrusts of the flying buttresses.

The flying buttresses of Fig. 46 are well-placed, and could perhaps be made slightly more slender and steeper, (like those of Lichfield, Fig. 23); the lower buttress, resisting a vault thrust of 20 ton, will be working in a definite active rather than a passive state.

The main external buttress presents no problem of design, and is adequately treated by, for example, Moseley. Viollet-le-Duc (article "Cathédrale") draws attention to the seeming disproportion in Fig. 46 of the lower and upper portions of the main buttress, reinforcing his contention that money ran short at about this stage of the work.

Figure 49 summarizes some probable forces acting on the main nave pier, and totalling some 270 ton. (Note the estimated 10-ton vault thrust from the aisle; the vertical load of 20 ton is applied eccentrically, by means of corbels, to the nave pier. Cf. Breymann's

Fig. 408, in Fig. 5 here.) Working again to a permissible stress of 600 lb/in^2, a load of 270 ton requires an area of about 1000 in^2, corresponding to a nave pier diameter of about 36 in., which is considerably more slender than those of Reims.

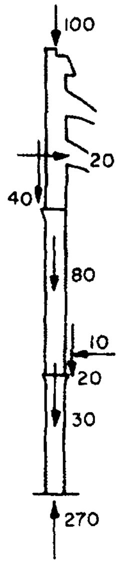

FIG. 49. Main pier, Reims analogue; loads in tons.

The slenderness ratio of a pier of height 36 ft and diameter 3 ft is $l/k = 36/0{\cdot}75 = 48$. Such a slenderness ratio seems well within permissible limits. For example, suppose that the slenderness ratio were fixed at $l/k = \pi E/\sigma_0$, where E is the elastic modulus and σ_0 the crushing strength of the stone; such a member would therefore crush before buckling as a pin-ended column. Taking E as low as seems likely at 3×10^6 lb/in^2 (see e.g. Breymann), and σ_0 at 6000 lb/in^2, l/k is determined as 70.

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Résumé—Le mode d'action de la construction en maçonnerie est étudié ici, en utilisant les principes de construction plastique développés à l'origine pour les cadres métalliques. Ces principes sont appliqués à l'analyse du système structural de la cathédrale gothique; l'arc-boutant et la voute de la nef sont étudiés en détail.

Zusammenfassung—Das Verhalten von Mauerwerkkonstruktion wird untersucht, unter Verwendung der Grundsätze des Traglastverfahrens welches ursprünglich für Stahlrahmen entwickelt wurden. Diese Grundsätze werden zur Analyse der Bauart des Gothischen Domes angewendet; die Strebebogen und das Domschiffgewölbe werden in aller Ausführlichkeit behandelt.

Абстракт—Исследуется образ действия каменной или кирпичной кладки, с употреблением принципов пластического расчета, разработанного первоначально для стальных конструкций. Эти принципы применяются для анализа структурной системы Готического собора. Подпорная арка (арочный контрфос) и свод корабля собора рассматриваются в некоторых деталях.